UCLA Electrical Engineering Department - *Communication Systems Laboratory*

On the Design of Arbitrarily Low-Rate Turbo Codes

Miguel Griot
Andres I. Vila Casado
Richard Wesel

1

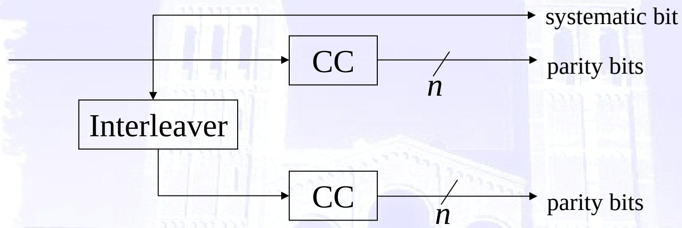
Low-rate codes: Motivation

- Practical:
  - Applications requiring very low SNRs: deep-space networks.
  - Spread-spectrum multiple-access channels: CDMA.
  - Interleave-Division Multiple-Access : IDMA [Li Ping'06].
  - Rateless coding.
- Theoretical:
  - Approach the ultimate low-rate Shannon limit of $E_b / N_0 \simeq -1.59dB$ on the AWGN Channel.

2

Low-rate turbo codes

- We consider low-rate turbo codes:

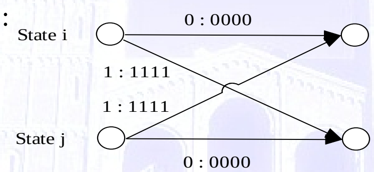


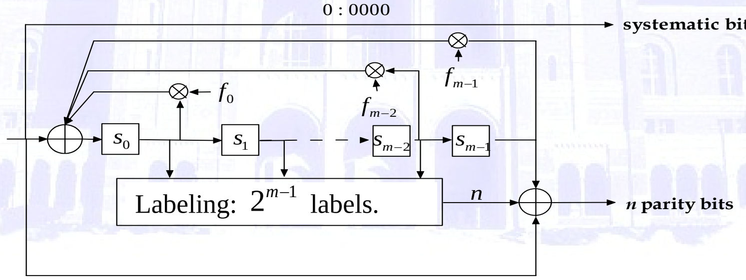
- Given a certain number of trellis states of each constituent encoder:
  - Goal: approach ultimate Shannon capacity $E_b / N_0 \simeq -1.59dB$.
  - What is the best rate? Structure?

3

Previous work

- Super-orthogonal turbo codes:
  - For a certain number of states of the constituent code 2^m and a certain trellis structure.
  - Use antipodal labeling on splits and merges (Ungerboeck's rules idea):

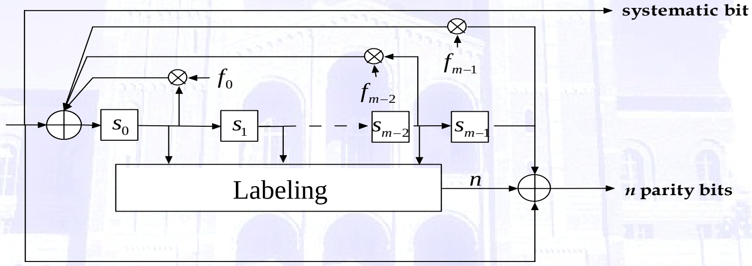




4

Questions

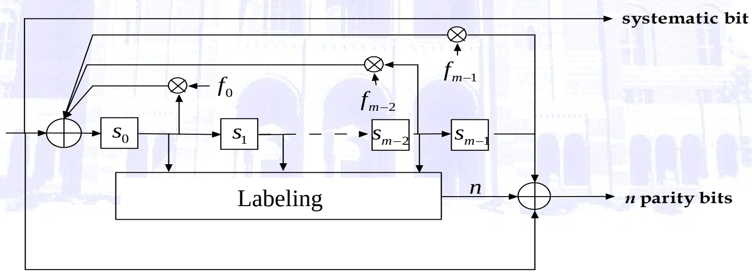
- To optimize performance vs. Eb/No:
  - Given a certain rate and number of states of the constituent encoder, what is the best structure and labeling?
  - What is the best set of rates?



General Constituent Encoder Structure

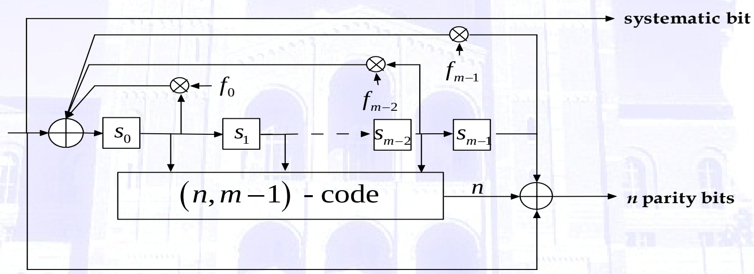
- Given a certain number of states 2^m then the recursive structure of the trellis can be chosen to maximize the length of trellis error events with input Hamming distance 2 (effective free distance). This number is 2^m.

m	$f_0 f_1 \cdots f_{m-1}$
2	11
3	011
4	0011
5	00101



Labeling

- Use antipodal output labeling in splits and merges.
- The output labeling design can be seen as a code design.



What is the optimal value of n?

- The larger n the larger the minimum distance of the code for labeling.
- We want to optimize the code in terms of BER vs. E_b / N_0
- Now for an (n,k) code and an AWGN channel:

$$P_e \approx Q\left(\sqrt{2d_{\min} E_s / N_0}\right) = Q\left(2(k/n)d_{\min} E_b / N_0\right)$$

- $k = m\text{-}1$ fixed.
- Criteria: maximize $\dfrac{d_{\min}}{n}$
- Although the rate of the turbo code is 1/(2n+1), the effect of the systematic bit has little effect on the performance of low-rate turbo codes.

BCH codes are optimal

- Theorem I: every (n,k) code must satisfy:

$$\frac{d_{\min}}{n} \le \frac{2^{k-1}}{2^k - 1}$$

- Proof for linear codes: linear codes produce equally likely ones and zeros. Consider the sum of the distances between the all-zero codeword and non-zero codewords.

$$\left(2^k - 1\right) d_{\min} \le \sum_{i=1}^{2^k-1} d_H\left(0, c_i\right)$$

$$\sum_{i=1}^{2^k-1} d_H\left(0, c_i\right) \le \sum_{i=0}^{2^k-1} W_H\left(c_i\right) = 2^k \left(\frac{n}{2}\right) = 2^{k-1} n$$

- For nonlinear codes with unequally likely ones and zeros, strict inequality.

9

BCH codes are optimal

- Theorem I: every (n,k) code must satisfy:

$$\frac{d_{\min}}{n} \le \frac{2^{k-1}}{2^k - 1}$$

- Theorem II: Equality holds when $n = 2^k - 1$ using a BCH code or when $n = r\left(2^k - 1\right)$ using a BCH code with an (r,1) repetition code.
- In that case $d_H\left(c_i, c_j\right) = d_{\min} = r \times 2^{k-1}$.
- This provides the labeling and the rate.
- Rate = $\dfrac{1}{1 + 2r\left(2^{m-1} - 1\right)}$

10

BCH codes

- BCH codes can be represented with generator polynomials. In this case, all $2^{m-2}-1$ possible polynomials of the form:

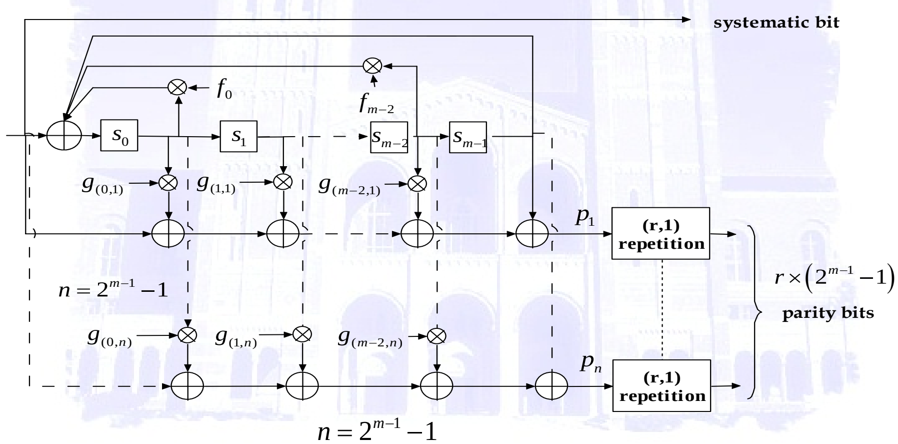
$$g(x) = a_0 + a_1 x + \cdots + a_{m-2} x^{m-2} \neq 0$$

- They can also be represented by the rows of a Hadamard matrix.

11

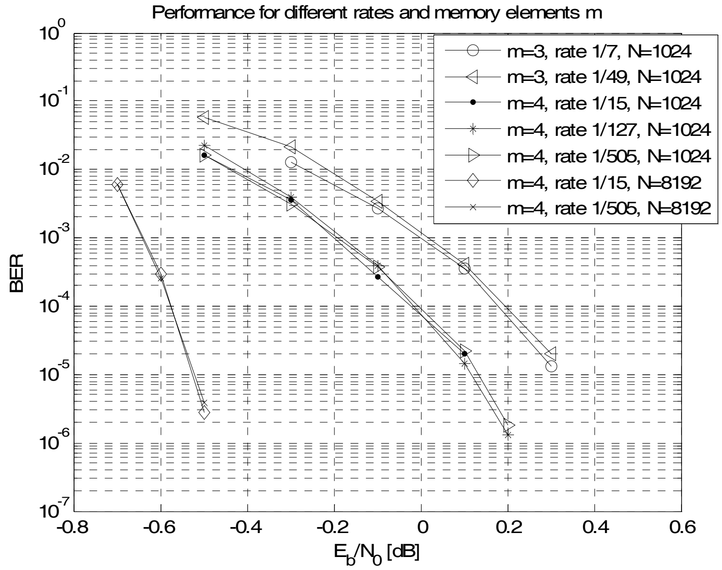
Labeling of Linear constituent encoder

- BCH codes can be represented with generator polynomials.



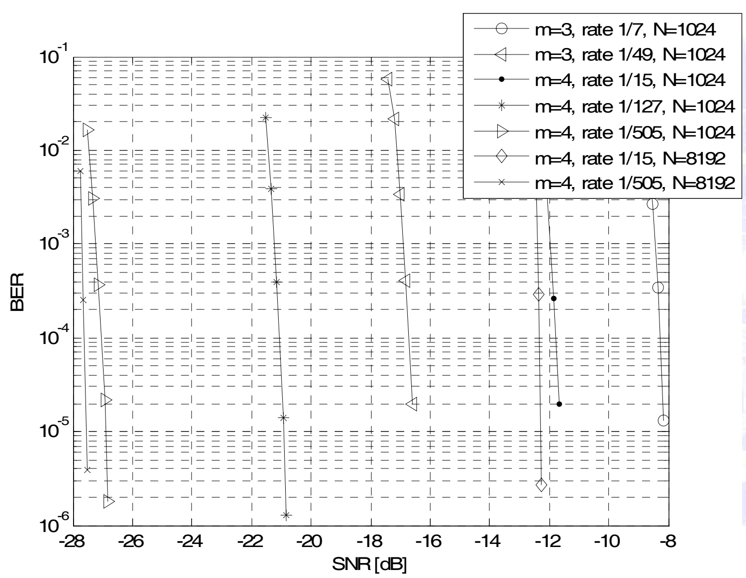
12

Performance: Eb/No



13

Performance: SNR



14

Conclusions

- We have shown a criteria to design low-rate turbo codes.
- Goal: Minimize BER vs. Eb/No.
- We see the output labeling problem as a coding problem.
- Criteria: maximize $\frac{d_{\min}}{n}$
  - Given the number of states 2^m, provides set of rates as well as structure and labeling,
  - $\left(2^{m-1}-1, m-1\right)$ BCH codes are optimal for labeling under this criteria.
  - $n = r\left(2^{m-1}-1\right)$

15