Iterative Equalization and Decoding

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COMSOC Distinguished Lecture Tour

Conventional Equalization

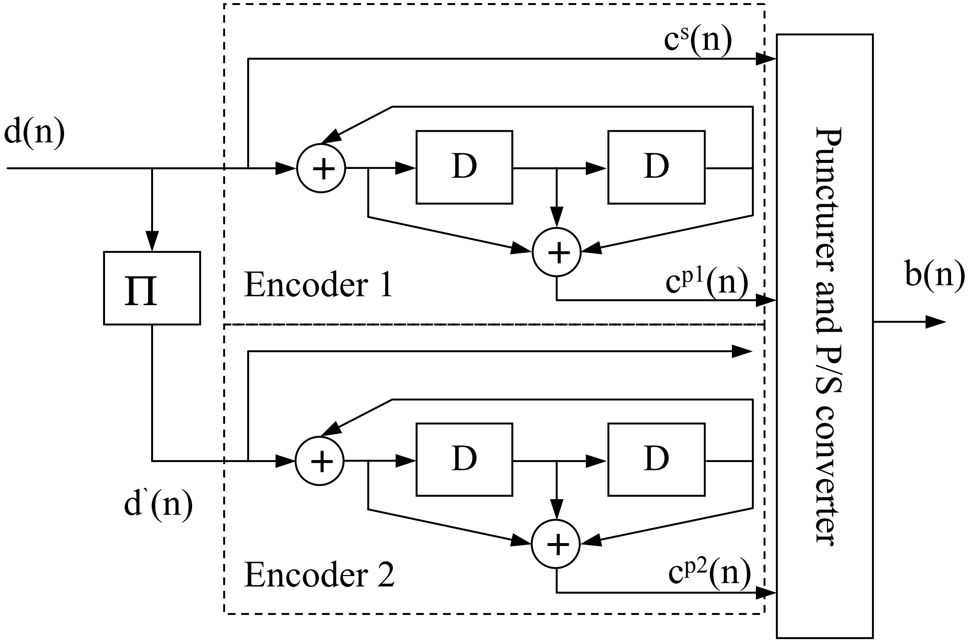
Possible Equalizer Types:

- Linear Equalizer
- Decision Feedback Equalizer (DFE)
- Maximum A posteriori Probability (MAP) Equalizer
- Soft-output Viterbi (MLSE) Equalizer

Possible Decoder Types:

- Maximum A posteriori Probability (MAP) Decoder
- Viterbi (MLSE) Decoder

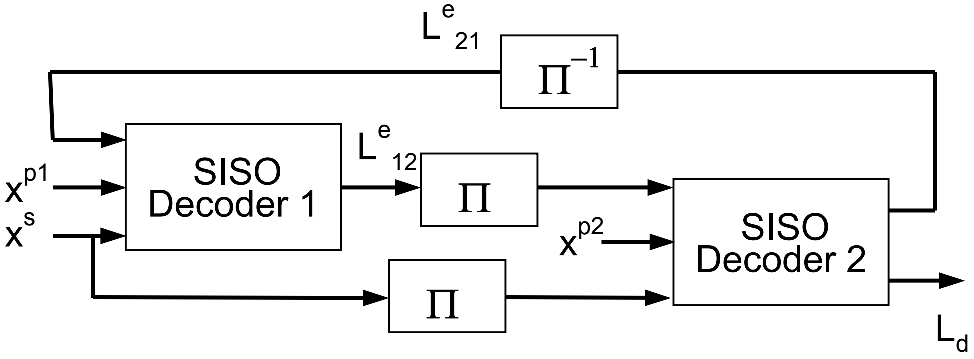
Turbo Principle / Turbo Coding



Turbo Encoder:

- *Parallel concatenated* recursive systematic convolutional encoders
- Encoders separated by an interleaver

Turbo Decoder:

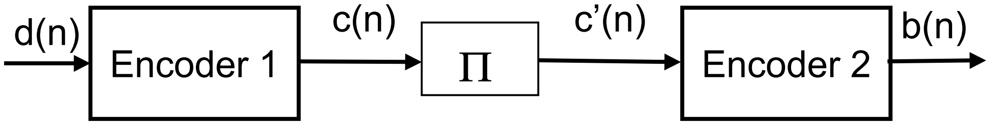


- Two Soft-Input Soft-Output (SISO) decoders separated by interleavers
- SISO modules can be
  - SOVA
  - MAP
- Extrinsic information passed between modules

Serially Concatenated Systems

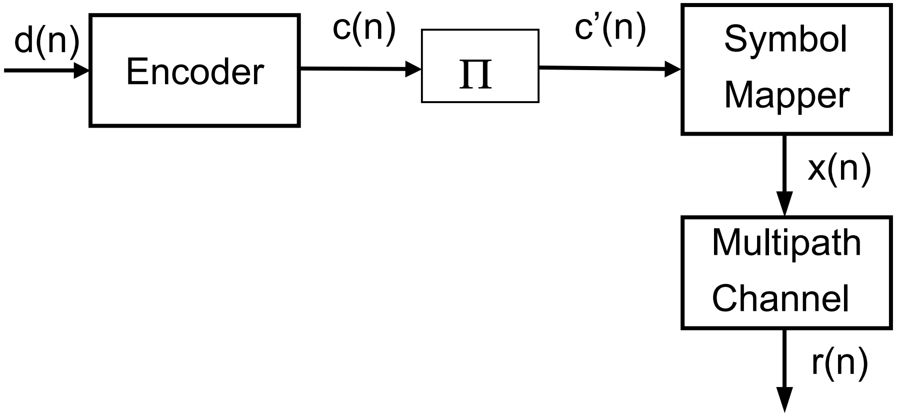
Serially Concatenated Coding:

- *Serial concatenated* (recursive) convolutional encoders
- Encoders separated by an interleaver



Coded Transmission over Multipath Channels:

- (recursive) convolutional encoder
- Interleaved bits mapped to symbols
- Symbols passed through a multipath channel



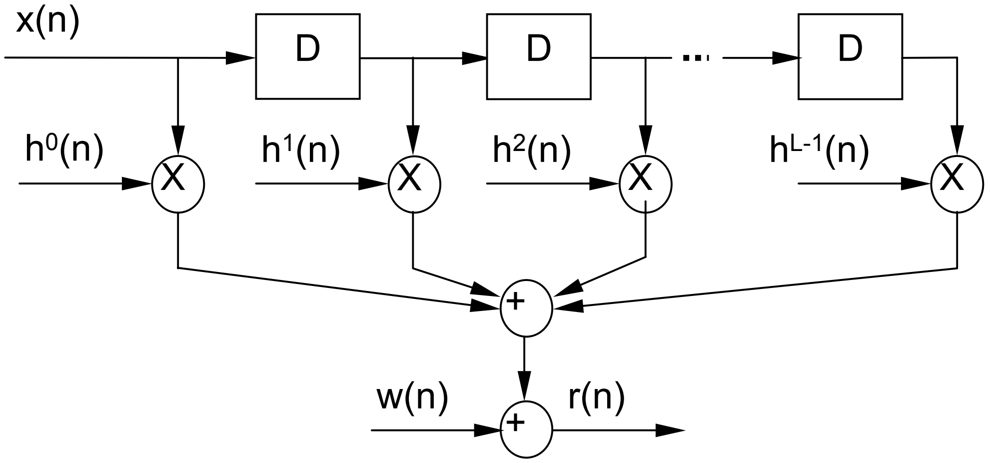
Channel Model / Precoding

Multipath Channel Model:

- Received signal:

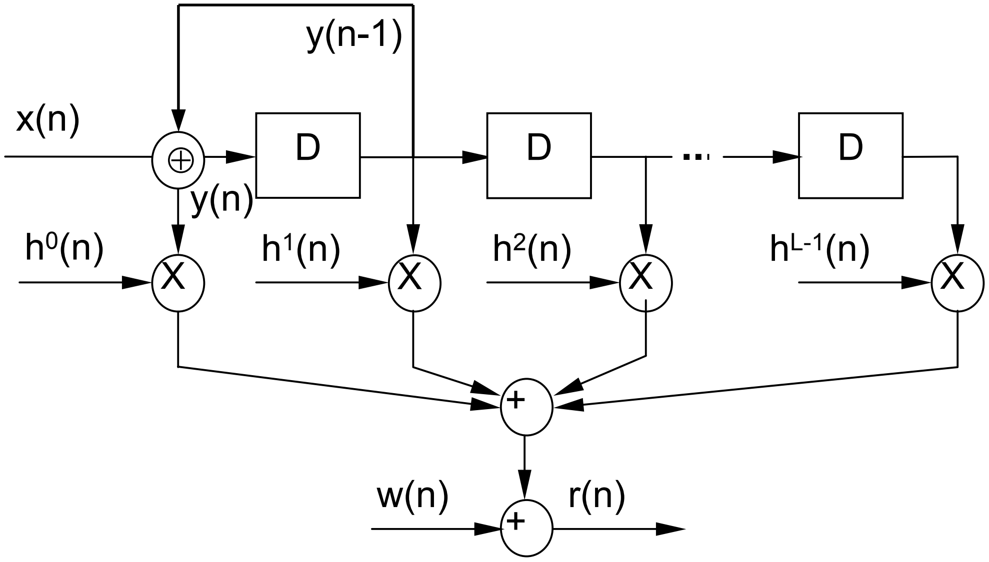
$$r(n) = \sum_{l=0}^{L} h^{l}(n)x(n-l) + w(n)$$

- Rate 1/1 convolutional code



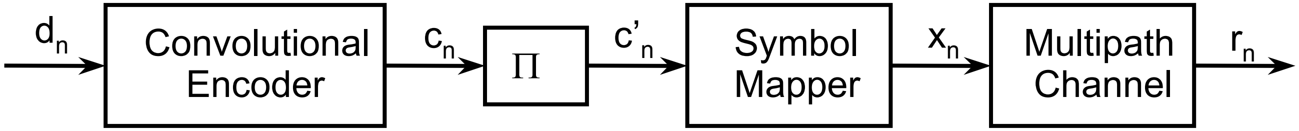
Precoded System:

- Iteration gain only possible with recursive inner code (channel) [1], [2]
- Recursive rate 1/1 precoder is employed before transmission
- Most common precoder: Differential encoder

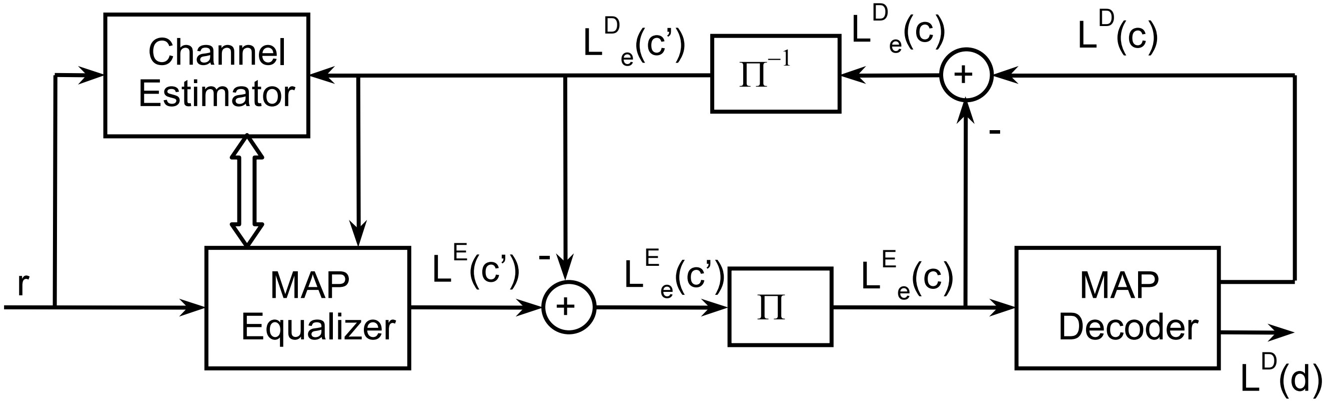


Iterative Equalization and Decoding (Turbo Equalizer)

- Data bits are convolutionally encoded and interleaved
- M-ary PSK modulated signals transmitted through a multipath channel, which is treated as an encoder

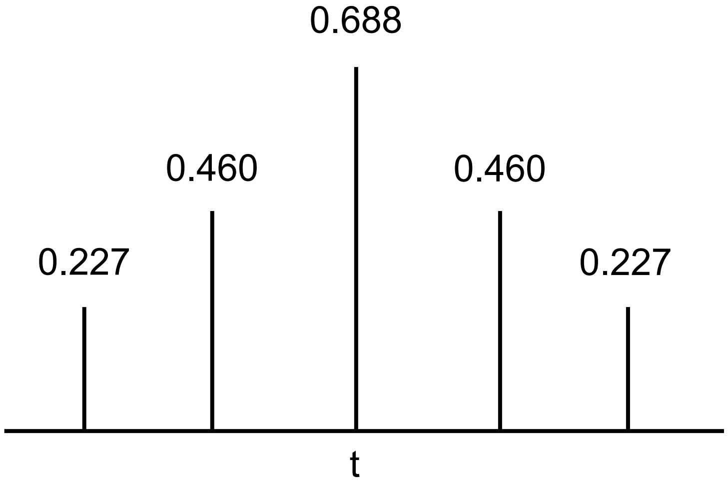


- Received signals are jointly equalized and decoded in a turbo structure
- First proposed by Douillard, *et.al* [3], where SOVA modules are employed
- Bauch extended the idea by employing MAP modules [4]

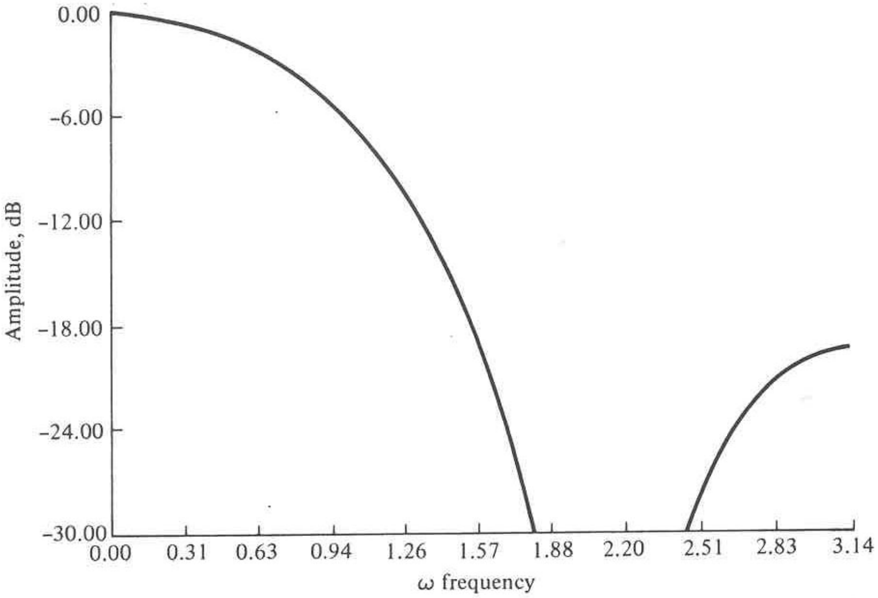


Time-invariant Test Channel

- Proakis C channel [5]



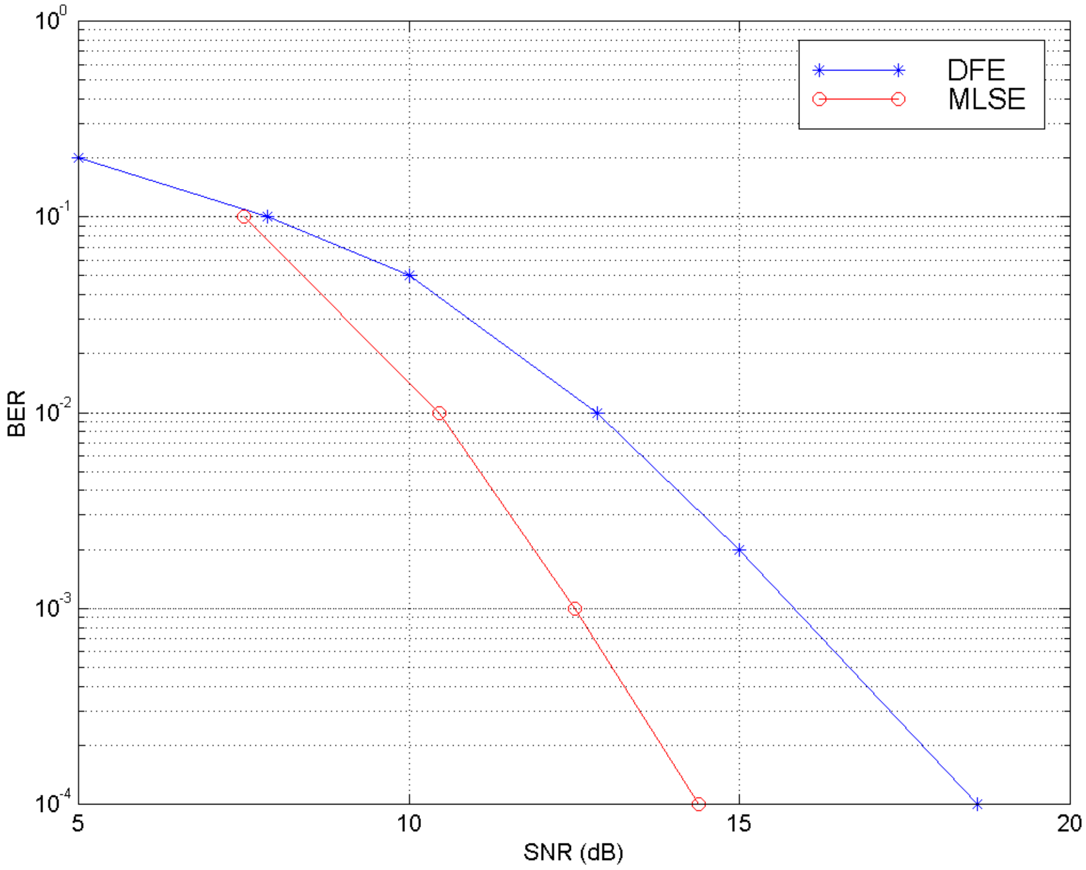
Impulse Response



Frequency Response

Low Complexity Alternative Equalizers: DFE and MLSE

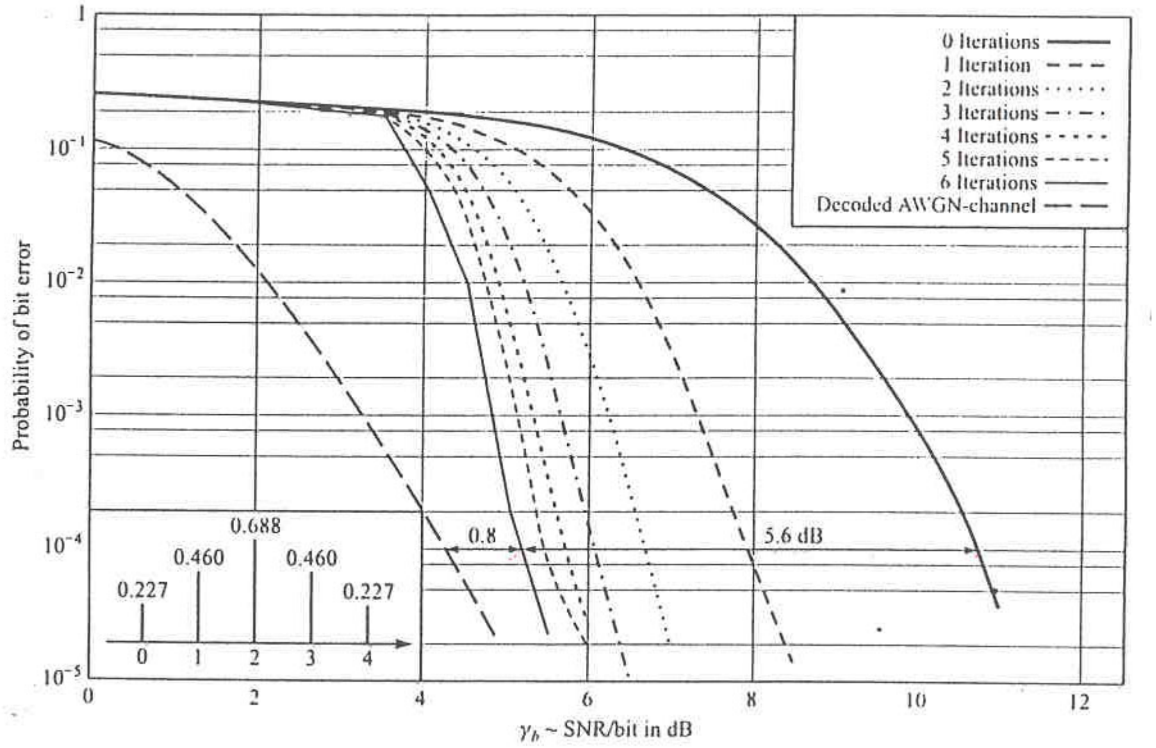
- Performance of DFE and MLSE over the Proakis C channel [5]



Bit error rate performance [5]

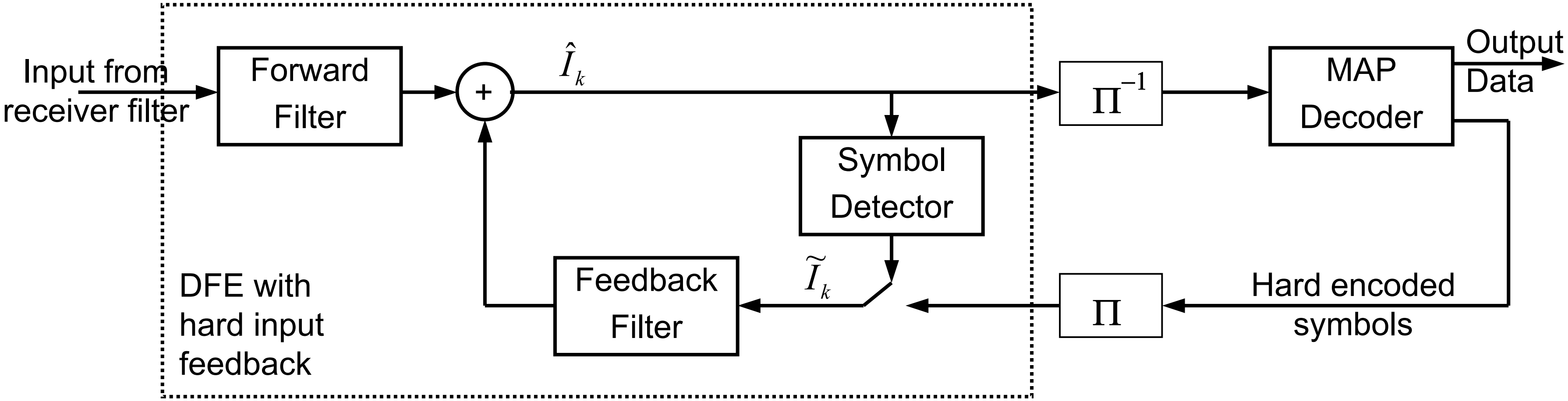
Iterative Equalization and Decoding Performance

- Iterative equalization and decoding with MAP modules
- Recursive systematic convolutional encoder with R=1/2, K=5,
- Time-invariant 5 tap channel with a spectral null (Proakis C [5])
- Equalizer has perfect knowledge of the channel
- Block length 4096



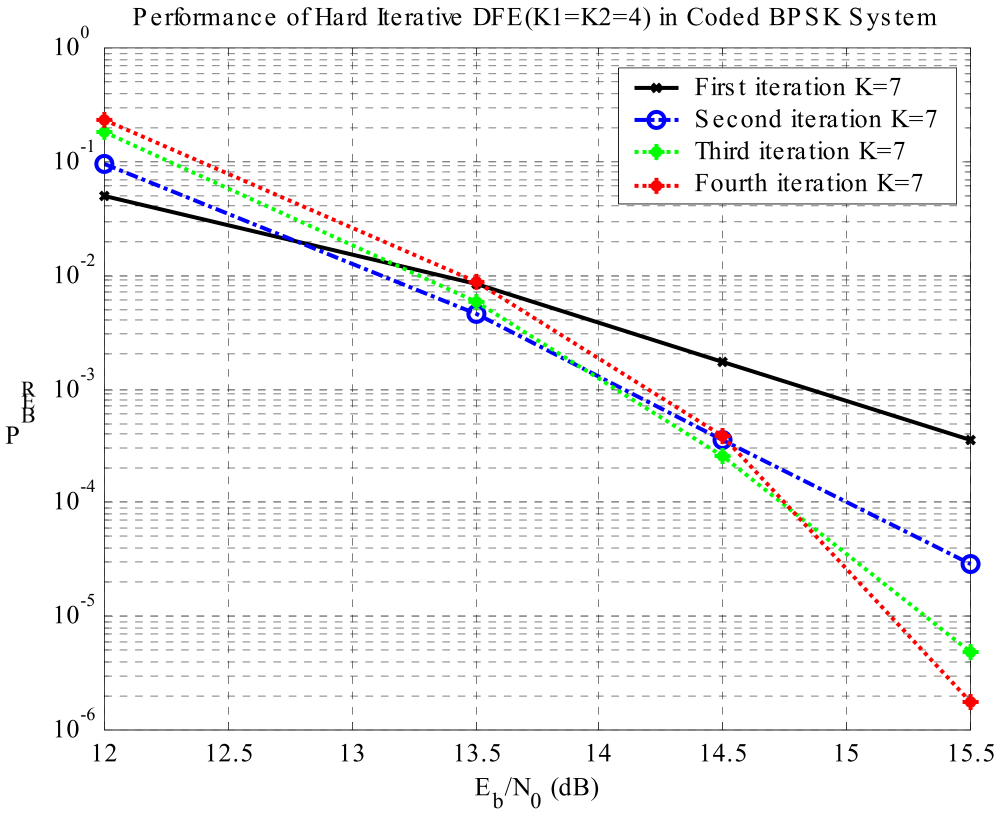
Bit error rate performance of turbo equalizer [4]

Hard Iterative DFE and MAP Decoding



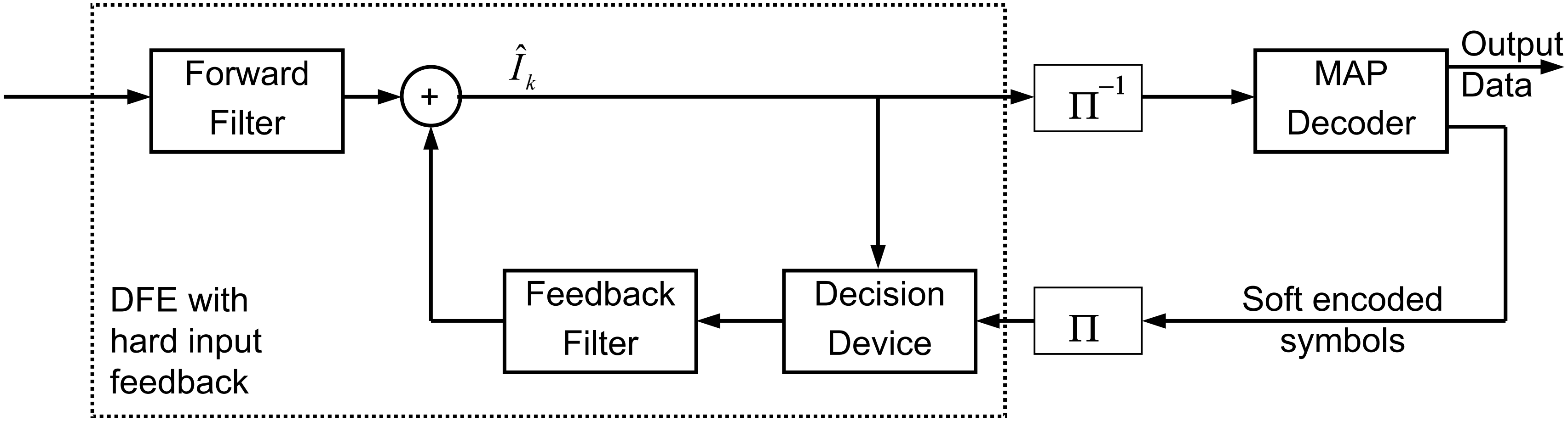
- During the first pass, symbol detector output is passed to the feedback filter
- After the first pass, hard encoded symbol output of the decoder is used in the feedback filter

Performance of Hard Iterative DFE and MAP decoder

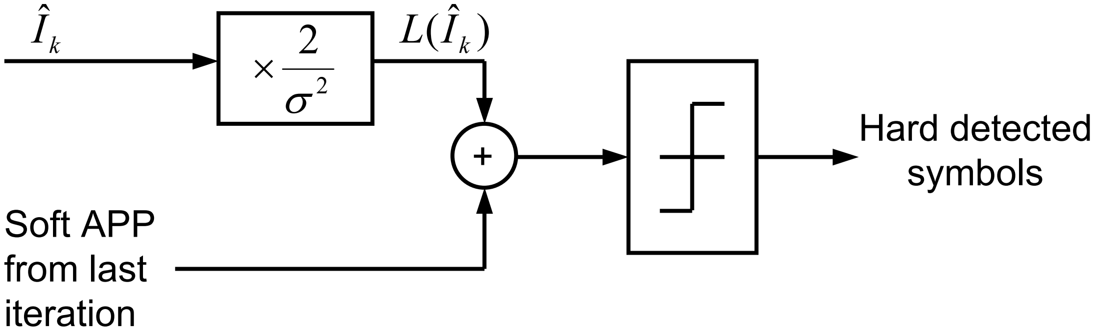


- BPSK modulation
- R=1/2, K=7 convolutional coding
- Block length 2048
- Channel Proakis C

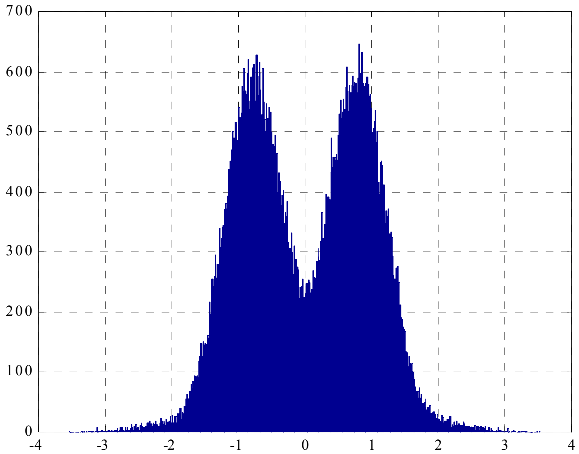
Soft Iterative DFE and MAP Decoding



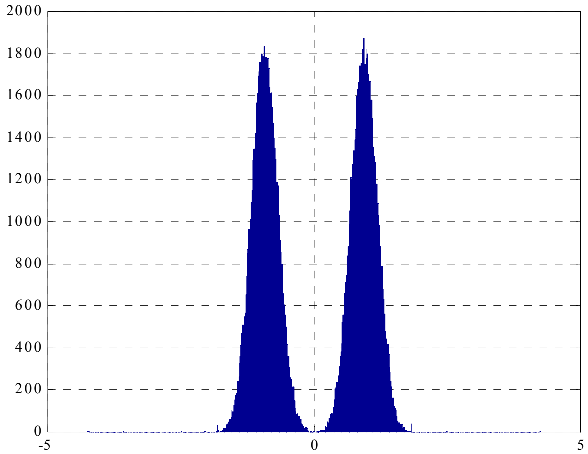
Soft decisions of the decoder is combined with the soft outputs of the DFE:



Histogram of DFE Output

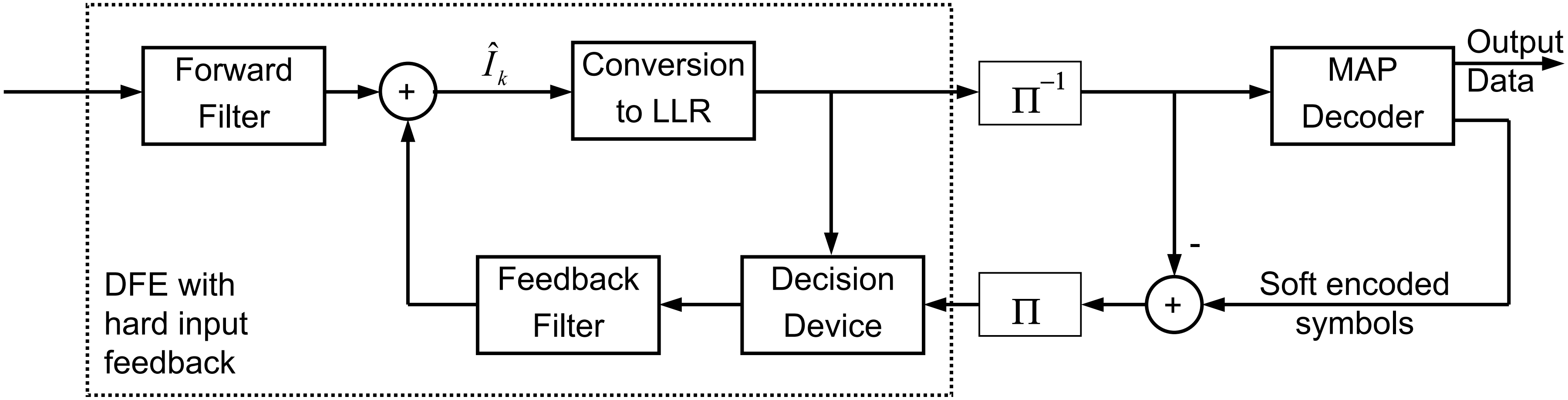


The histogram of equalizer estimated output for SNR = 12 dB

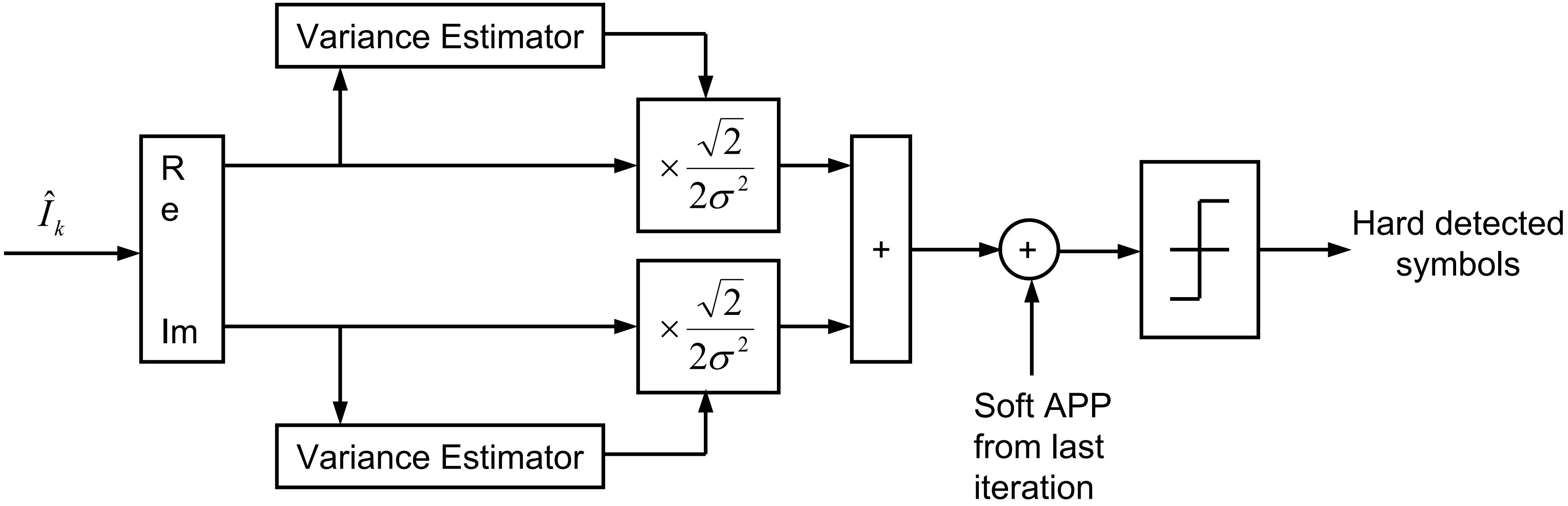


The histogram of equalizer estimated output for SNR = 20 dB

Modified Soft Iterative DFE and MAP Decoding

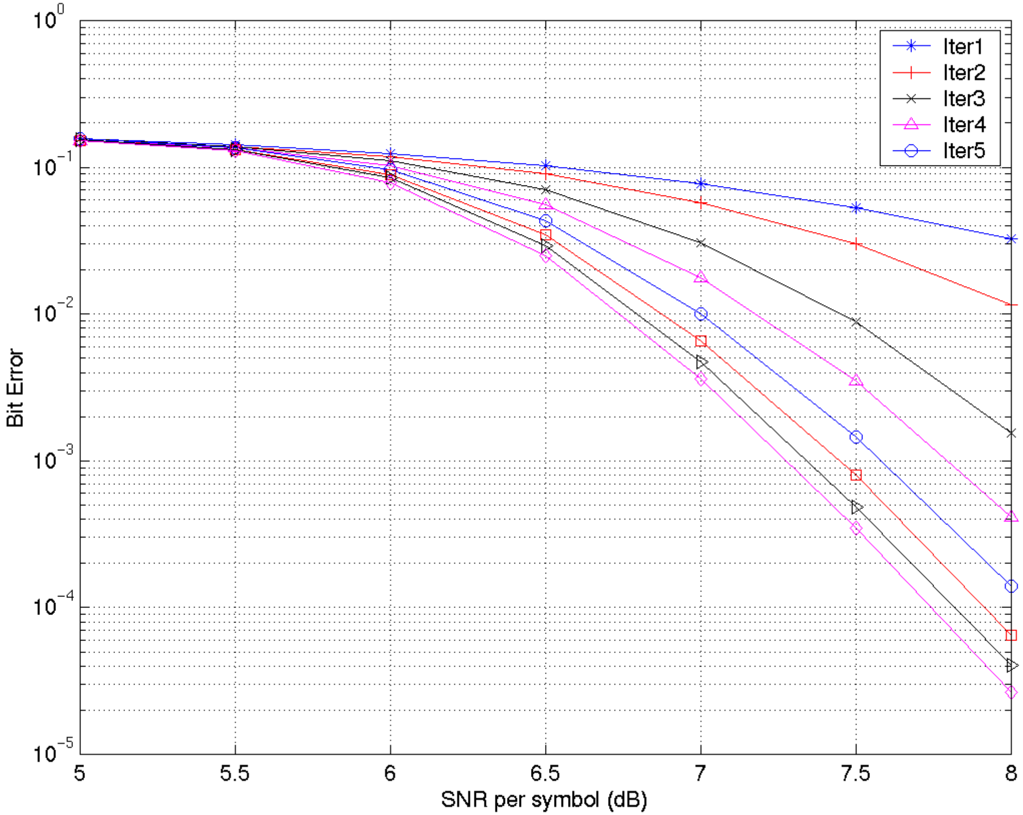


Only extrinsic information is passed to the DFE from the decoder



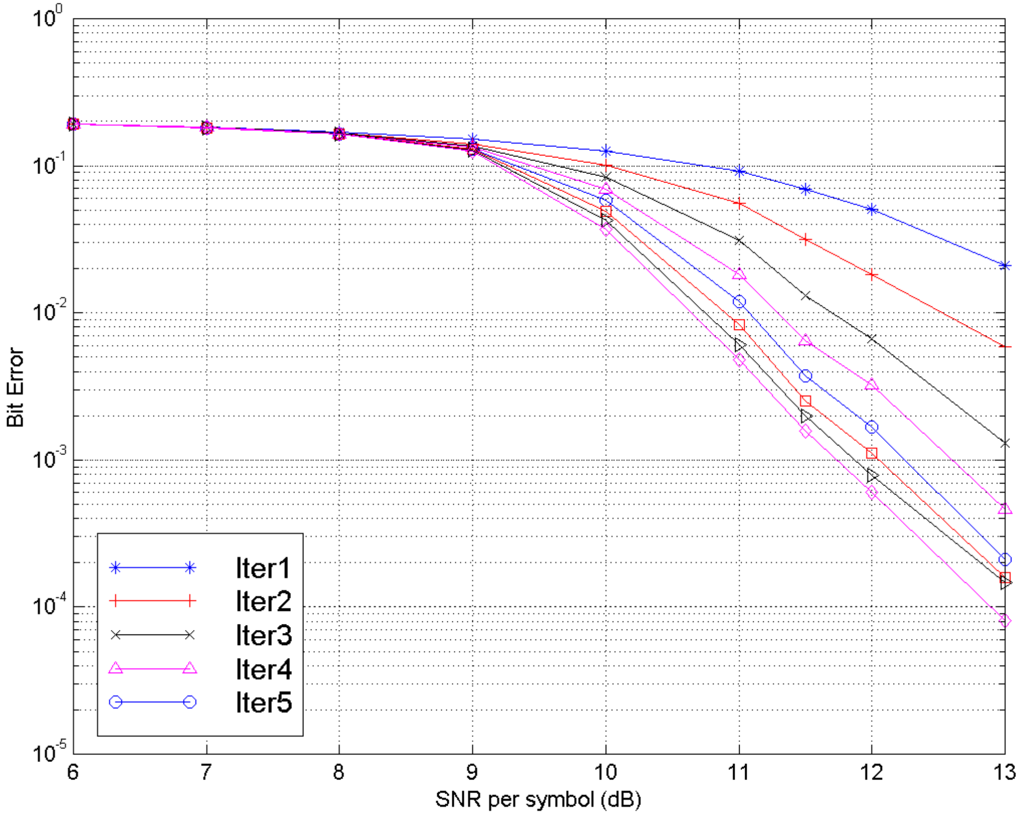
Performance of Soft Iterative DFE and MAP Decoder

- Recursive systematic convolutional encoder with R=1/2, K=5,
- Time-invariant 5 tap channel with a spectral null (Proakis C [5])
- RLS updates at the DFE
- Block length 4096
- BPSK modulation



Performance of Soft Iterative DFE and MAP Decoder

- Recursive systematic convolutional encoder with R=1/2, K=5,
- Time-invariant 5 tap channel with a spectral null (Proakis C [5])
- RLS updates at the DFE
- Block length 4096
- QPSK modulation



Iterative Linear MMSE Equalization and Decoding

SISO Linear MMSE Equalizer [6]:

- Known channel: $h_k, k = -M_1, -M_1 + 1, \cdots, M_2$
- Received signal: $z_n = \sum_{k=-M_1}^{M_2} h_k x_{n-k} + \omega_k$
- Likelihood ratio for MMSE estimator output, $\hat{x}_n$: $L^{MMSE}(x_n) = \underbrace{\ln \frac{p(\hat{x}_n \mid x_n = +1)}{p(\hat{x}_n \mid x_n = -1)}}_{L_e^{MMSE}(x_n)} + \underbrace{\ln \frac{\Pr\{x_n = +1\}}{\Pr\{x_n = -1\}}}_{L(x_n)}$
- Channel matrix: $\mathbf{H} = \begin{bmatrix} h_{-M_1} & h_{-M_1+1} & \cdots & h_{M_2} & 0 & \cdots & 0 \\ 0 & h_{-M_1} & h_{-M_1+1} & \cdots & h_{M_2} & \cdots & 0 \\ & & & \ddots & & & \\ 0 & \cdots & 0 & h_{-M_1} & h_{-M_1+1} & \cdots & h_{M_2} \end{bmatrix}$
- MMSE estimator output: $\hat{x}_n = \mathbf{c}_n^H \left(\mathbf{z}_n - \mathbf{m}_n^z\right) + m_n^x$

where,

$$\mathbf{c}_n = \left(\sigma_\omega^2 \mathbf{I}^N + \mathbf{H}\,\mathrm{cov}(\mathbf{x}_n, \mathbf{x}_n)\mathbf{H}^H\right)^{-1} \mathrm{cov}(\mathbf{x}_n, x_n)$$

$$\mathbf{m}_n^z = E\{\mathbf{z}_n\} = \mathbf{H}E\{\mathbf{x}_n\}$$

$$m_n^x = E\{x_n\} = \tanh\left(\frac{1}{2}L(x_n)\right)$$

Iterative Linear MMSE Equalization and Decoding

Steps to compute symbol estimates with the Linear MMSE equalizer:

$$m_n^x = \tanh\left(\frac{1}{2}L(x_n)\right)$$

$$\mathbf{m}_n^x = \left[m_{n+M_1+N_1}^x \quad m_{n+M_1+N_1-1}^x \quad \cdots \quad m_{n-M_2-N_2}^x\right]^T$$

$$\mathbf{D}_n = diag\left(1-(m_{n+M_1+N_1}^x)^2 \quad 1-(m_{n+M_1+N_1-1}^x)^2 \quad \cdots \quad 1-(m_{n-M_2-N_2}^x)^2\right)$$

$$\mathbf{s} = \mathbf{H}\left[0_{1x(N_1+M_1)} \quad 1 \quad 0_{1x(N_2+M_2)}\right]^T$$

$$\mathbf{c}_n = \left(\sigma_\omega^2 \mathbf{I}^N + \mathbf{H}\mathbf{D}_n\mathbf{H}^H + (m_n^x)^2\mathbf{s}\mathbf{s}^{\mathbf{H}}\right)^{-1}\mathbf{s}$$

$$\hat{x}_n = \mathbf{c}_n^H\left(\mathbf{z}_n - \mathbf{H}\mathbf{m}_n^x + m_n^x\mathbf{s}\right)$$

Soft output calculation assuming Gaussian distributed estimates:

$$\mu_n^{(+1)} = E\{\hat{x}_n \mid x_n = +1\} = \mathbf{c}_n^H\mathbf{s}$$

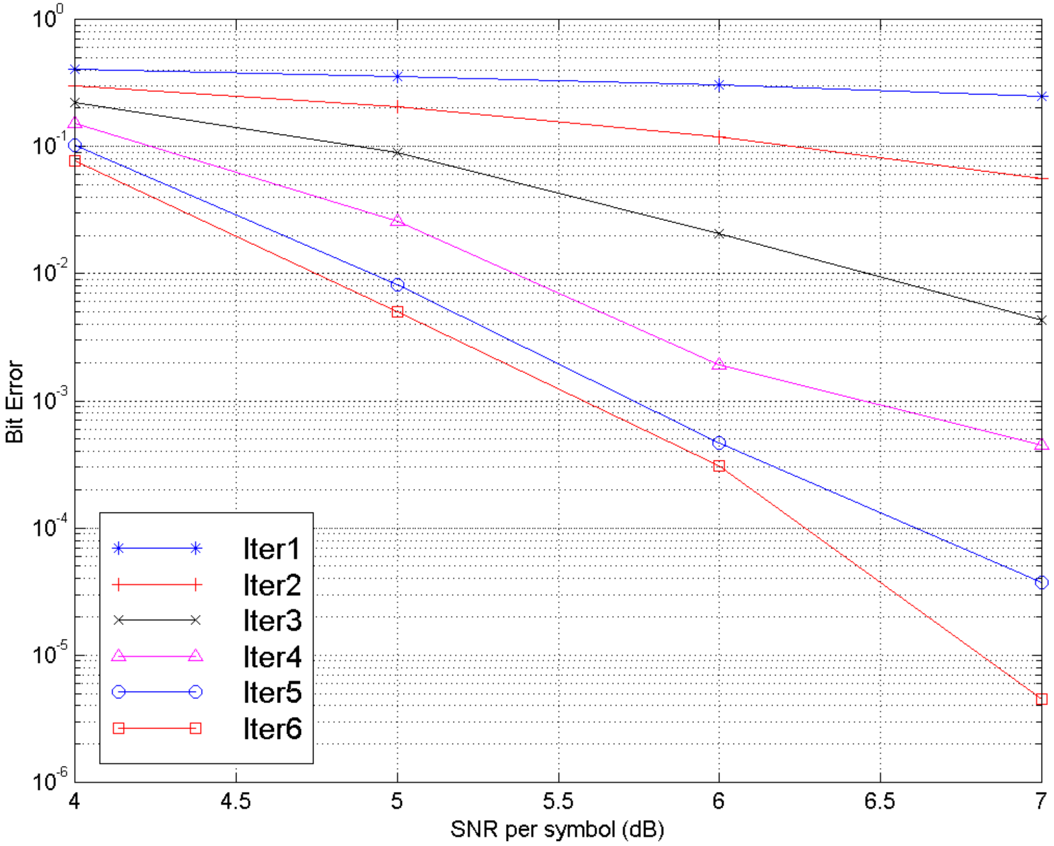
$$\mu_n^{(-1)} = E\{\hat{x}_n \mid x_n = -1\} = -\mathbf{c}_n^H\mathbf{s}$$

$$\sigma_n^2 = \mathbf{c}_n^H\mathbf{s} - \left|\mu_n^{(+1)}\right|^2$$

$$L_e^{MMSE}(x_n) = \frac{2\hat{x}_n\mu_n^{(+1)}}{\sigma_n^2} = \frac{2\hat{x}_n}{1-\mathbf{s}^H\mathbf{c}_n}$$

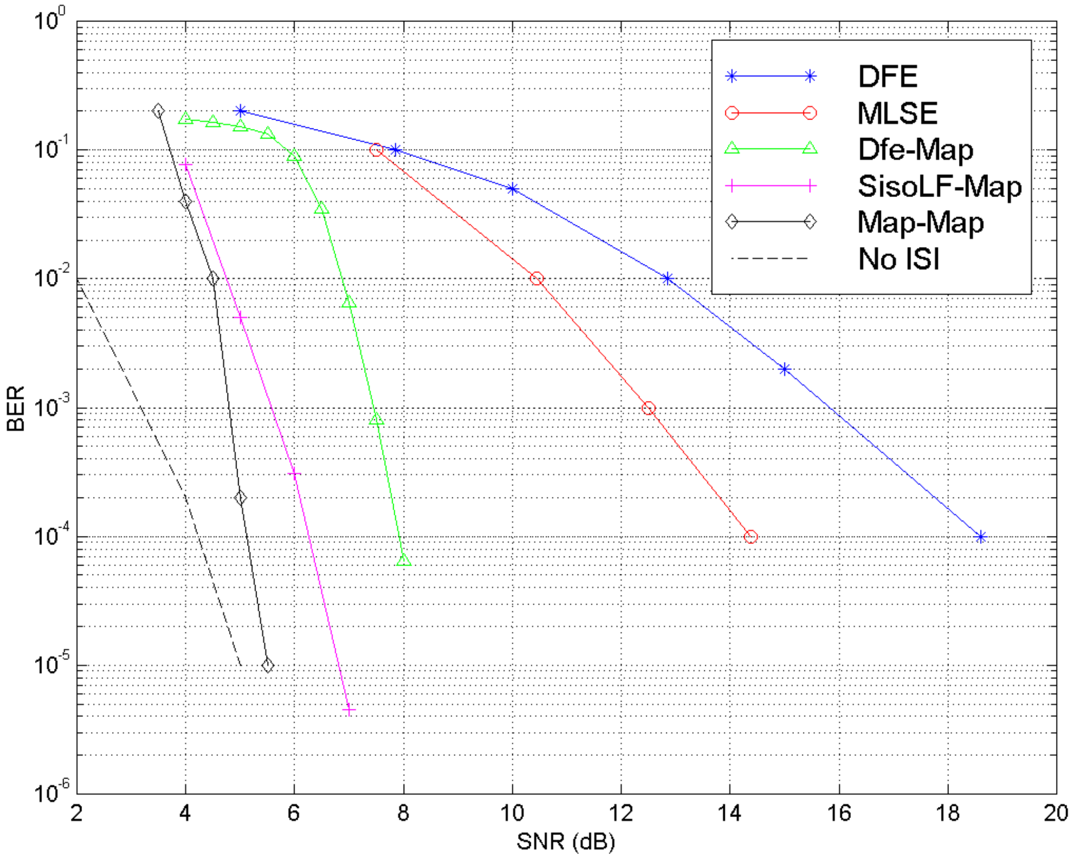
Performance of SISO MMSE Linear Iterative Equalizer

- Recursive systematic convolutional encoder with R=1/2, K=5,
- Time-invariant 5 tap channel with a spectral null (Proakis C [5])
- Equalizer has perfect knowledge of the channel
- Block length 4096

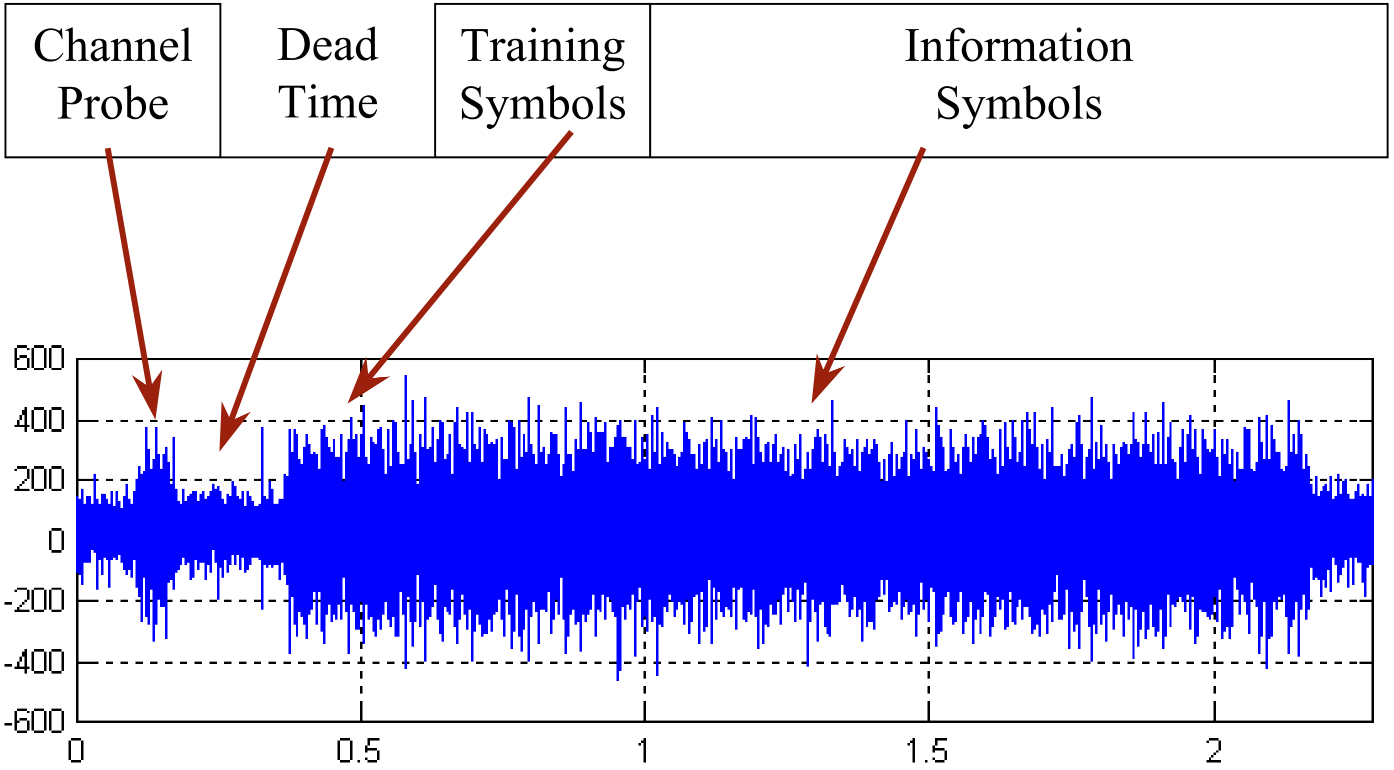


Comparison of System Performances

- Recursive systematic convolutional encoder with R=1/2, K=5,
- Time-invariant 5 tap channel with a spectral null (Proakis C [5])
- BER results after 6 iterations
- Block length 4096



Experimental Study of Iterative Equalizers



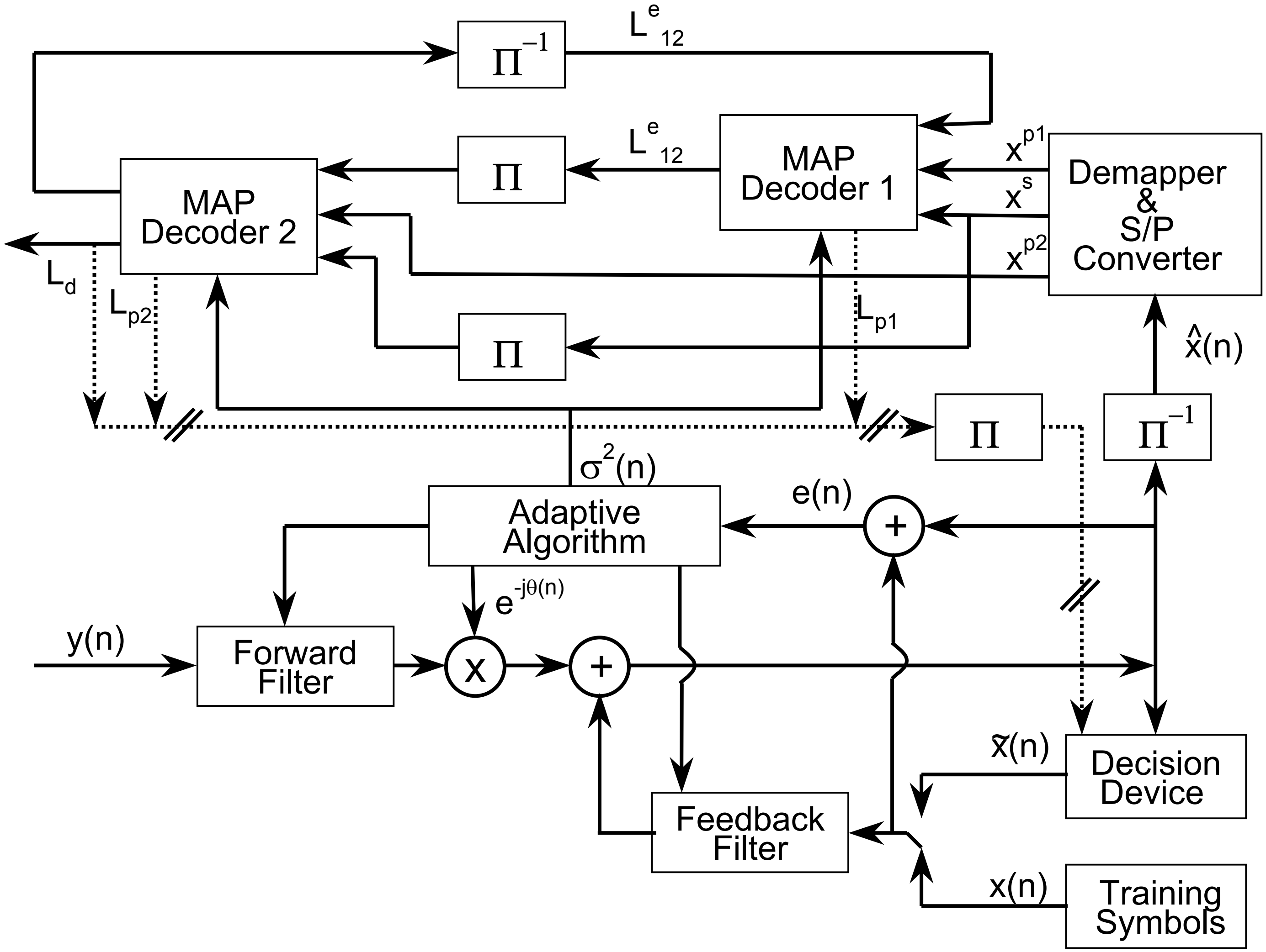
Channel Probe

Dead Time

Training Symbols

Information Symbols

Joint MMSE Equalization and Turbo Decoding



Decision Feedback Equalizer (DFE)

- Soft output of the DFE: $\hat{x}(n) = \mathbf{w}_{\mathbf{f}}^{H}(n)\mathbf{y}(n) - \mathbf{w}_{\mathbf{b}}^{H}(n)\widetilde{\mathbf{x}}(n)$

•RLS algorithm is used to track channel variation:

$$\mathbf{k}(n) = \frac{\mathbf{P}(n-1)\mathbf{y}(n)}{\lambda + \mathbf{y}^{H}(n)\mathbf{P}(n-1)\mathbf{y}(n)}$$

$$\mathbf{P}(n) = \lambda^{-1}(\mathbf{P}(n-1) - \mathbf{k}(n)\mathbf{y}^{H}(n)\mathbf{P}(n-1))$$

$$\xi(n) = x(n) - (\mathbf{w}_{\mathbf{f}}^{H}(n-1)\mathbf{y}(n) - \mathbf{w}_{\mathbf{b}}^{H}(n-1)\widetilde{\mathbf{x}}(n))$$

$$\mathbf{w}_{\mathbf{f}}(n) = \mathbf{w}_{\mathbf{f}}(n-1) + \mathbf{P}(n)\mathbf{y}(n)\xi^{*}(n)$$

$$\mathbf{w}_{\mathbf{b}}(n) = \mathbf{w}_{\mathbf{b}}(n-1) + \mathbf{P}(n)\widetilde{\mathbf{x}}(n)\xi^{*}(n)$$

- Noise variance estimate:

$$\sigma^{2}(n) = \lambda\sigma^{2}(n-1) + (1-\lambda)\left|x(n) - (\mathbf{w}_{\mathbf{f}}^{H}(n)\mathbf{y}(n) - \mathbf{w}_{\mathbf{b}}^{H}(n)\widetilde{\mathbf{x}}(n))\right|^{2}$$

MAP Decoding

- Maximize a posteriori probability:

$$P(d_n = d_i \mid \mathbf{y}), \quad i = 0,1,\cdots,k-1$$

- Decision variable written in the form of log-likelihood ratio:

$$\Lambda(dn) = \log\left(\frac{P(d_n = +1 \mid \mathbf{y})}{P(d_n = -1 \mid \mathbf{y})}\right)$$

$$= \log\left(\frac{\sum_{S^+} p(s_{n-1}=s', s_n=s, \mathbf{y}) / p(\mathbf{y})}{\sum_{S^-} p(s_{n-1}=s', s_n=s, \mathbf{y}) / p(\mathbf{y})}\right)$$

MAP Decoding (BCJR Algorithm)

- State transition probability:

$$\gamma_{\mathrm{n}}(s',s) = P(d_n)\,p(\bar{y}_n \mid \bar{x}_n)$$

where

$$\bar{y}_n = (y_n^s, y_n^p) \qquad \bar{x}_n = (x_n^s, x_n^p)$$

$$\Rightarrow \quad \gamma_{\mathrm{n}}(s',s) = \exp\left(d_n\left(\frac{L^e(d_n)}{2} + \frac{y_n^s}{\sigma^2}\right)\right)\exp\left(\frac{y_n^p x_n^p}{\sigma^2}\right)$$

$$\Rightarrow \quad \Lambda(d_n) = \frac{2}{\sigma^2}y_n + L^e(d_n) + \log\left(\frac{\sum_{S^+}\alpha_{n-1}(s')\gamma_n^e(s',s)\beta_n(s)}{\sum_{S^-}\alpha_{n-1}(s')\gamma_n^e(s',s)\beta_n(s)}\right)$$

channel value

a priori information

extrinsic information

MAP Equalizer

- $\gamma_{\mathrm{n}}^{\mathrm{e}}(s', s)$ depends on the channel trellis defined by $h^{l}(n)$ with $2^{(L-1)}$ states

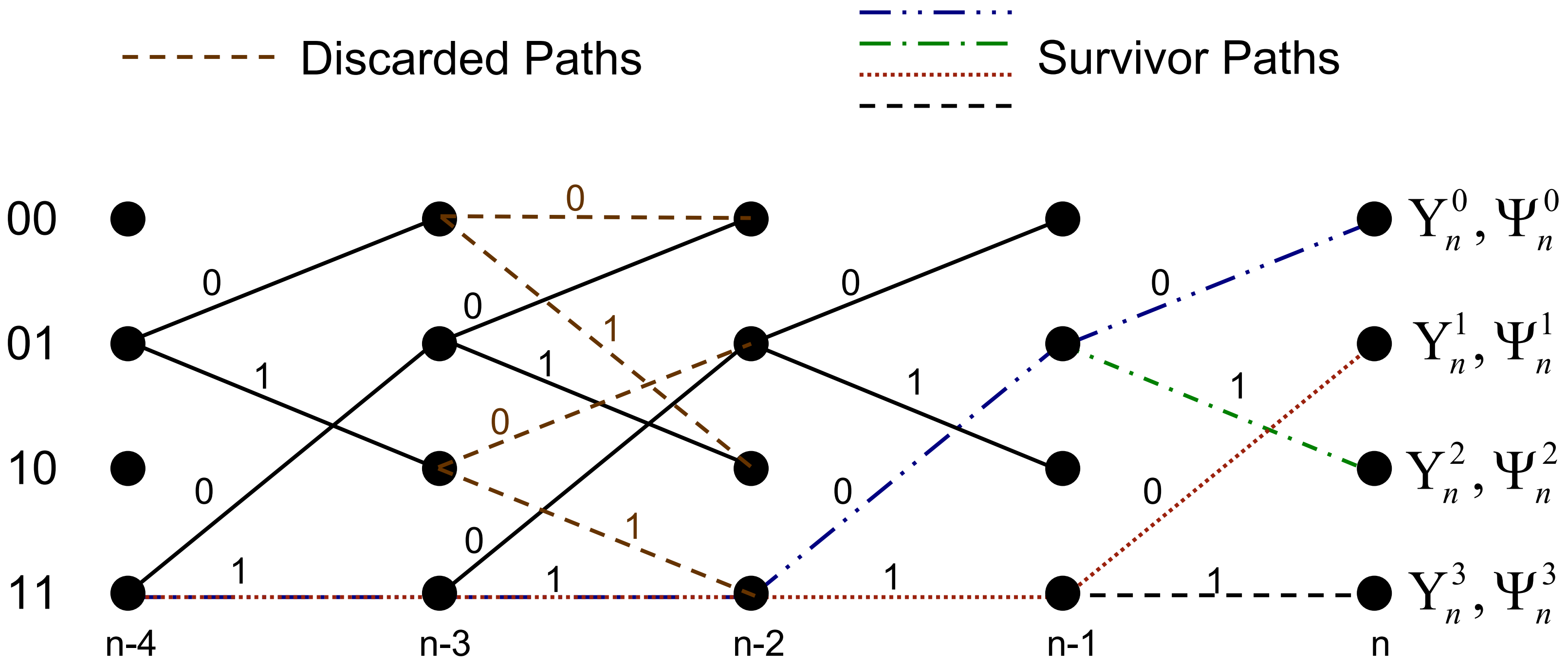
$$\gamma_{\mathrm{n}}^{\mathrm{e}}(s', s) = \exp\left(-\frac{1}{2\sigma^{2}} \left| y_{n} - \sum_{l=0}^{L-1} h^{l}(n) x_{n-l} \right| \right)$$

- If x_{n-l} for $J<l<L-1$ is known

$$\gamma_{\mathrm{n}}^{\mathrm{e}}(s', s) = \exp\left(-\frac{1}{2\sigma^{2}} \left| y_{n} - \sum_{l=0}^{J-1} h^{l}(n) x_{n-l} - \sum_{l=J}^{L-1} h^{l}(n) \hat{x}_{n-l} \right| \right)$$

Number of states is reduced to $2^{(J-1)}$

Per-Survivor Processing

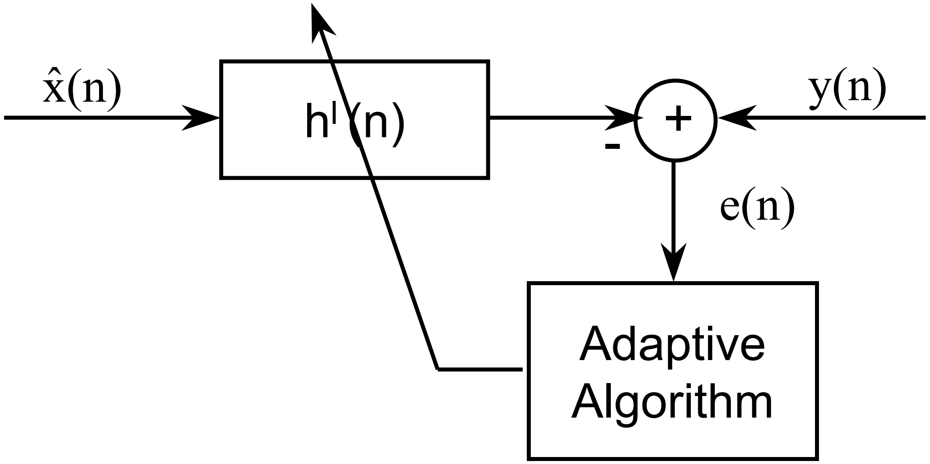


Path metric: $\Psi_n^{S_m} = P\left(Y_n^{S_m}\right) = \prod_n \alpha_n\left(S_m \mid Y_n^{S_m}\right)$

Survivor path: $Y_n^0 = [1,1,0,0]$

$Y_n^2 = [1,1,0,1]$

Channel Estimator



- Each survivor path has a separate channel estimator
- The input to the channel estimator, $\hat{x}(n)$, is the estimates within the survivors
- RLS algorithm is employed

Channel Estimator

- Initial channel estimate is based on the correlation of the preamble
- RLS algorithm is employed to track the channel

$$\mathbf{k}(n) = \frac{\mathbf{P}(n-1)\hat{\mathbf{x}}(n)}{\lambda + \hat{\mathbf{x}}^H(n)\mathbf{P}(n-1)\hat{\mathbf{x}}(n)}$$

$$\mathbf{P}(n) = \lambda^{-1}(\mathbf{P}(n-1) - \mathbf{k}(n)\hat{\mathbf{x}}^H(n)\mathbf{P}(n-1))$$

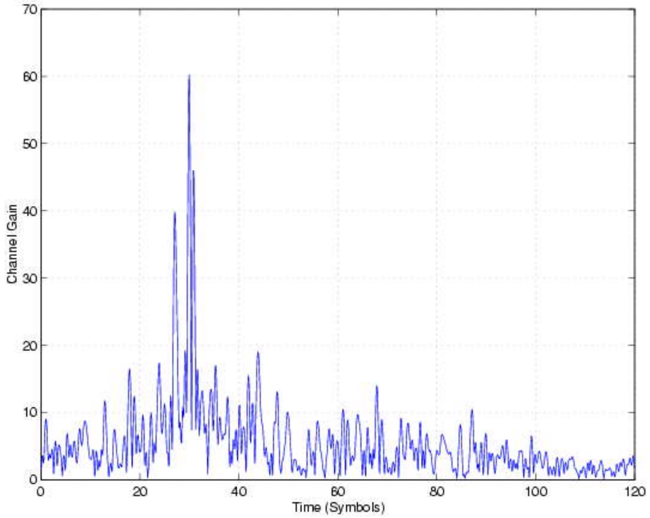
$$\xi(n) = y(n) - (\mathbf{w}^H(n-1)\hat{\mathbf{x}}(n))$$

$$\mathbf{w}(n) = \mathbf{w}(n-1) + \mathbf{P}(n)\hat{\mathbf{x}}(n)\xi^*(n)$$

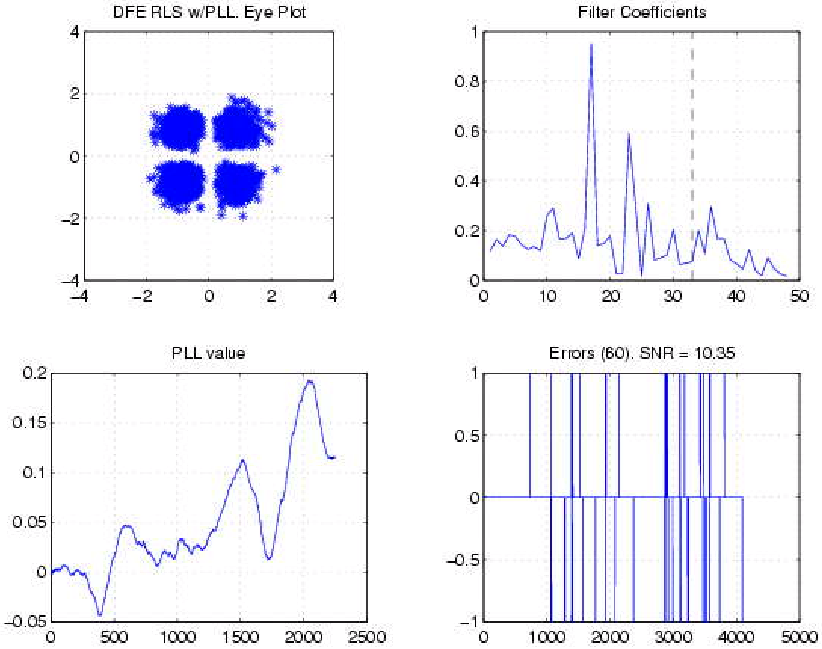
- Noise variance estimate:

$$\sigma^2(n) = \lambda\sigma^2(n-1) + (1-\lambda)\left|\xi(n)\right|^2$$

Experimental Results

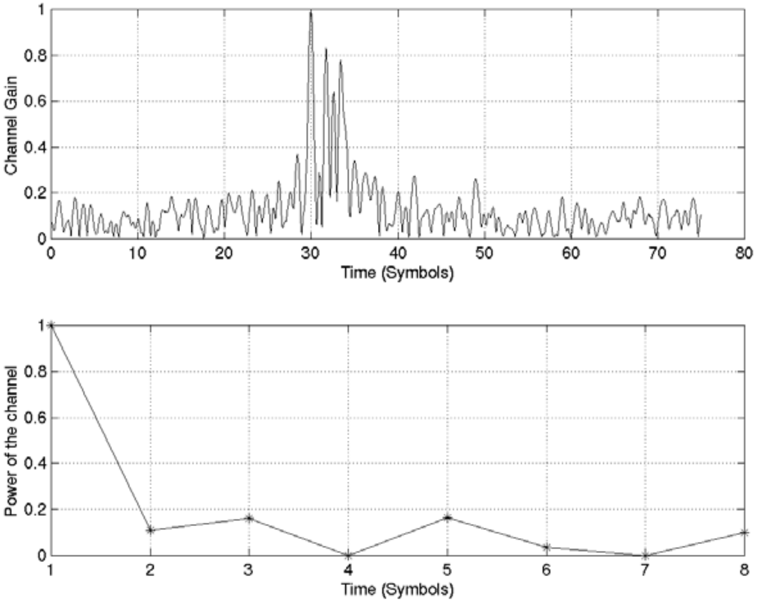


Channel impulse response estimate for transducer seven obtained using the channel probe

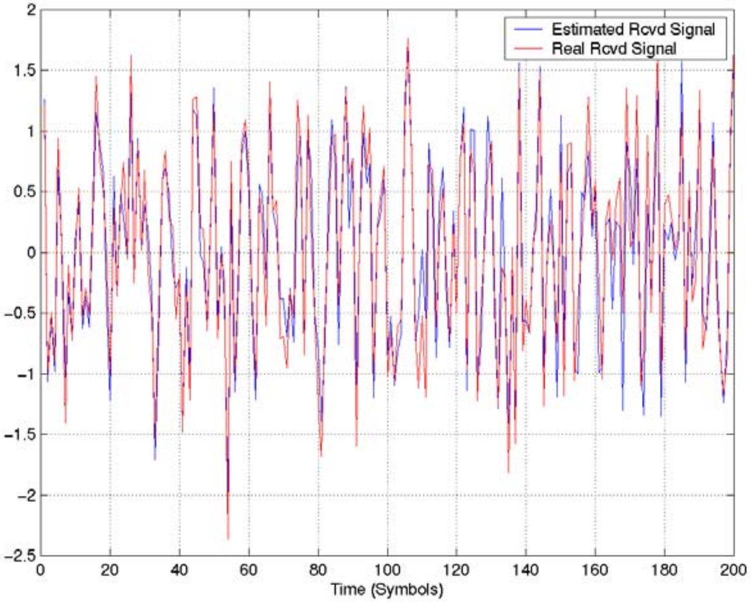


DFE results for transducer 7.
Eye Pattern - Filter coefficients
PLL phase estimate - Bit error distribution

Experimental Results

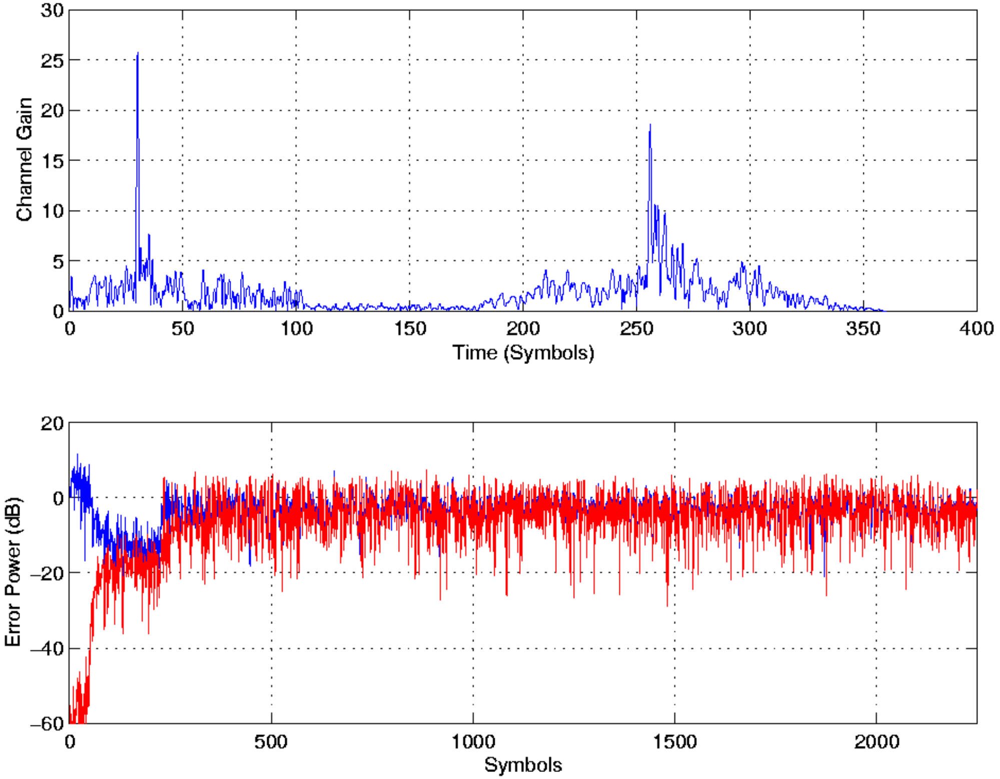


Channel impulse response estimate for transducer seven obtained using adaptive channel estimator



Comparison of received signal with the estimated received signal based on the channel estimate

Experimental Results



- Sparse Channel with multipath delay in the order of 200 symbols
- Length of the DFE or channel estimator filters cannot cover the channel
- Sparse processing is needed

Results of DFE Turbo Decoder

Event	Modulation	DFE	Iter 1	Iter 2	Iter 3	Iter 4	Iter 5
0	QPSK	334	25	4	3	3	3
	BPSK	70	0				
1	QPSK	488	80	56	49	47	49
	BPSK	22	0				
2	QPSK	392	43	18	15	15	15
	BPSK	64	4	2	2		
3	QPSK	374	27	17	12	11	11
	BPSK	51	0				
4	QPSK	290	11	7	7	7	7
	BPSK	38	0				
5	QPSK	121	2	0			
	BPSK	20	0				
6	QPSK	165	1	2	2	2	2
	BPSK	7	0				
7	QPSK	247	6	2	2	2	2
	BPSK	1	0				
8	QPSK	207	7	1			
	BPSK	17	1	0			

Results of Iterative DFE Turbo Decoder

Event	Mod.	DFE	Iter1	Iter2	DFE	Iter1	Iter2	DFE	Iter1	Iter2	DFE	Iter1	Iter2	DFE	Iter1	Iter2
0	QPSK	334	25	4	140	1	2	133	1	1	138	1	1	139	1	1
1	QPSK	488	80	56	203	16	9	179	8	5	171	7	4	172	5	4
2	QPSK	392	43	18	186	7	7	177	6	6	173	6	6	173	6	6
3	QPSK	374	27	17	165	5	6	148	2	0						
4	QPSK	290	11	7	139	5	5	131	3	3	129	3	3	129	3	3
5	QPSK	121	2	0												
6	QPSK	165	1	2	73	1	1	75	2	0						
7	QPSK	247	6	2	104	0										
8	QPSK	207	7	1	89	0										

Results of Iterative DFE MAP Decoder

Event	Mod.	DFE1	MAP 1	DFE2	MAP 2	DFE3	MAP 3	DFE4	MAP 4	DFE5	MAP 5	DFE6	MAP 6
0	QPSK	512	136	268	49	215	31	212	31	210	31	210	31/16
	BPSK	46	0										
1	QPSK	307	0										
	BPSK	40	0										
2	QPSK	522	167	265	13	210	0						
	BPSK	42	0										
3	QPSK	288	40	155	5	144	5	146	5	146	5	146	5/2
	BPSK	47	0										
4	QPSK	238	5	119	0								
	BPSK	36	0										
5	QPSK	153	0										
	BPSK	10	0										
6	QPSK	208	9	81	0								
	BPSK	6	0										
7	QPSK	91	0										
	BPSK	7	0										
8	QPSK	282	15	109	5	101	0						
	BPSK	16	0										

Results for Iterative Map Equalizer Turbo Decoder

PK #	MOD.	Equ Map 1	Tur Iter 1	Tur Iter 2	Equ Map 2	Tur Iter 1	Tur Iter 2	Equ Map 3	Tur Iter 1	Tur Iter 2	Equ Map 4	Tur Iter 1	Tur Iter 2	Equ Map 5	Tur Iter 1	Tur Iter 2
0	BPSK	21	0													
0	QPSK	447	73	50	80	3	3	8	0							
1	BPSK	7	0													
1	QPSK	682	241	222	444	154	115	197	38	25	33	4	1	3	3	0
2	BPSK	8	0													
2	QPSK	427	64	45	76	1	3	6	0							
3	BPSK	14	0													
3	QPSK	392	59	35	62	2	1	2	0							
4	BPSK	16	0													
4	QPSK	313	14	6	10	0										
5	BPSK	0														
5	QPSK	34	1	0												
6	BPSK	2	0													
6	QPSK	81	1	1	3	0										
7	BPSK	0														
7	QPSK	52	0													
8	BPSK	4	0													
8	QPSK	130	2	2	6	1	1	4	1	1	4	1	1	4	1	1

Results for Iterative MAP Equalizer MAP Decoder

PK #	MOD.	Equ MAP 1	Dec MAP 1	Equ MAP 2	Dec MAP 2	Equ MAP 3	Dec MAP 3
0	BPSK	13	0				
0	QPSK	600	333	212	38	29	0
1	BPSK	10	0				
1	QPSK	407	88	33	0		
2	BPSK	7	0				
2	QPSK	643	415	313	95	32	0
3	BPSK	20	0				
3	QPSK	294	9	6	0		
4	BPSK	18	0				
4	QPSK	327	34	19	0		
5	BPSK	1	0				
5	QPSK	85	0				
6	BPSK	8	0				
6	QPSK	113	0				
7	BPSK	2	0				
7	QPSK	77	0				
8	BPSK	2	0				
8	QPSK	277	14	1	0		

Conclusions

- Due to error propagation in the DFE, turbo decoder cannot provide performance improvement beyond the second iteration ⇒ Error Floor

- Joint DFE and turbo decoding adds an additional loop to the system and lowers the error floor

- Joint channel estimator and iterative equalizer is able to decode packets with low SNR, which cannot be decoded with the DFE

- Tail cancellation is an effective way to reduce the computational complexity of the MAP equalizer

- If the channel is sparse, although the DFE filter lengths are short, the DFE is able to provide enough information to the turbo decoder

- A sparse DFE can be used to improve the performance of the DFE/MAP Decoder and the DFE/Turbo Decoder

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