Markov Modeling

Markov modeling is a powerful tool used to simulate many apparently random multi-state processes. This method uses a state space approach to model the sequence of random or stochastic processes that govern the events occurring in the power system. In order to apply a Markov model to a power system reliability analysis every possible state and the transitions between them have to be explicitly modeled. For power system modeling the state transition matrix can become quite large. The computational burden of inverting the state transition matrix usually limits the applicability of Markov models to relatively small systems such as a single substation.

Determining the state probabilities at any time requires solving the Chapman-Kolmogorov differential equation, which can be created directly from the state space diagram and is defined

$$\mathbf{p}'(t)=\mathbf{p}(t)\mathbf{Q}$$

where $\mathbf{p}'(t)$ is the row vector of derivatives of the state probabilities with respect to time,

$$\mathbf{p}'(t) = [\ dp_1/dt \quad dp_2/dt \quad \ldots \quad dp_n/dt\ ]$$

$\mathbf{p}(t)$ is the row vector of state probabilities at time t,

$$\mathbf{p}(t) = [p_1 \quad p_2 \quad \ldots \quad p_n]$$

and $\mathbf{Q}$ is the state transition matrix whose elements are defined by

$$q_{ij} = \lambda_{ij}, \qquad i \neq j$$

and

$$q_{ii} = -\sum_{j \neq i} \lambda_{ij}\ , \ i = j$$

where λ_{ij} is the transition rate from state i to state j.

As time goes to infinity the state probabilities will reach a steady-state value. At steady-state all derivatives are equal to zero. Therefore, the steady-state probabilities can be found by a direct solution to

$$\mathbf{0}^T = \mathbf{pQ}$$

which can be written as a linear system of equations, i.e.

$$\mathbf{0} = \mathbf{Q}^T\mathbf{p}^T$$

The previous system of equations has only n-1 independent equations. To solve for the steady-state probabilities an additional equation is needed. That equation is

$$\sum_{i} p_i = 1$$

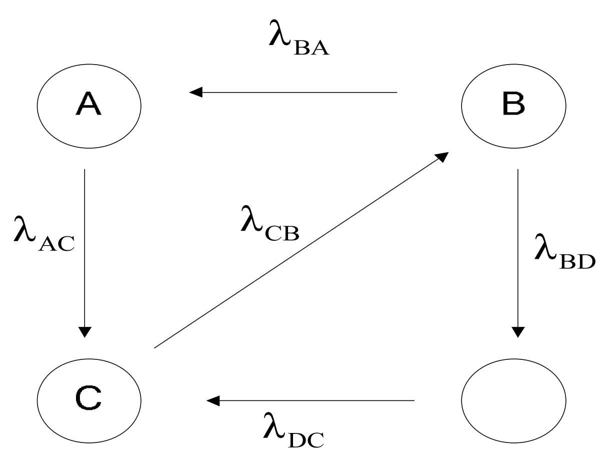
In addition to the probability of being in a given state the mean frequency at which states are encountered and the mean duration of stays within those states may also need to be computed in order to determine the system reliability indices. The frequency of encountering state *i*, f_i, is defined by

$$f_i = p_i \sum_{j \neq i} \lambda_{ij}$$

The mean duration of a stay in state *i*, T_i, is defined as the reciprocal of the total rate of departure from that state, i.e.

$$T_i = \frac{1}{\sum_{j \neq i} \lambda_{ij}}$$

To illustrate how these equations are applied the state transition matrix will be created for the following state space diagram.



To create the state transition matrix each diagonal in the matrix was chosen to correspond to a specific state. After choosing which diagonal represents which state, the transitions that exist between states were filled in and the diagonal elements of the matrix were then calculated. The complete state transition matrix is

$$\mathbf{Q} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{from} \rightarrow \text{to state} \\ \text{state} \end{array}$$

$$\mathbf{Q} = \begin{matrix} & A & B & C & D \\ A & -\lambda_{AC} & 0 & \lambda_{AC} & 0 \\ B & \lambda_{BA} & -(\lambda_{BA}+\lambda_{BD}) & 0 & \lambda_{BD} \\ C & 0 & \lambda_{CB} & -\lambda_{CB} & 0 \\ D & 0 & 0 & \lambda_{DC} & -\lambda_{DC} \end{matrix}$$

The Chapman-Kolmogorov equation for steady-state is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} p_A & p_B & p_C & p_D \end{bmatrix} \begin{bmatrix} -\lambda_{AC} & 0 & \lambda_{AC} & 0 \\ \lambda_{BA} & -(\lambda_{BA}+\lambda_{BD}) & 0 & \lambda_{BD} \\ 0 & \lambda_{CB} & -\lambda_{CB} & 0 \\ 0 & 0 & \lambda_{DC} & -\lambda_{DC} \end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\lambda_{AC} & \lambda_{BA} & 0 & 0 \\ 0 & -(\lambda_{BA}+\lambda_{BD}) & \lambda_{CB} & 0 \\ \lambda_{AC} & 0 & -\lambda_{CB} & \lambda_{DC} \\ 0 & \lambda_{BD} & 0 & -\lambda_{DC} \end{bmatrix} \begin{bmatrix} p_A \\ p_B \\ p_C \\ p_D \end{bmatrix}$$

The linear system of *n* equations has only *n-1* independent equations. Therefore, one equation in has to be replaced with

$$p_A + p_B + p_C + p_D = 1$$

The resulting set of linear equations is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\lambda_{AC} & \lambda_{BA} & 0 & 0 \\ 0 & -(\lambda_{BA}+\lambda_{BD}) & \lambda_{CB} & 0 \\ \lambda_{AC} & 0 & -\lambda_{CB} & \lambda_{DC} \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} p_A \\ p_B \\ p_C \\ p_D \end{bmatrix}$$

Any direct or iterative methods for solving linear equations can be used to solve for the steady-state probabilities. For most power systems applications the state space becomes quite large. Fortunately, the system of equations are diagonally dominant and very sparse. These properties can be exploited to compute the steady-state probabilities efficiently.

Once the state probabilities are known the probability, frequency and mean duration of failures can be determined for each load point. To do this the state space must be partitioned into failure and success domains for each load point. This partitioning is done by conducting a failure effects analysis to determine which states correspond to customer interruptions for each load point. Fig 3.2 shows a state space that has been partitioned into working and failed domains.

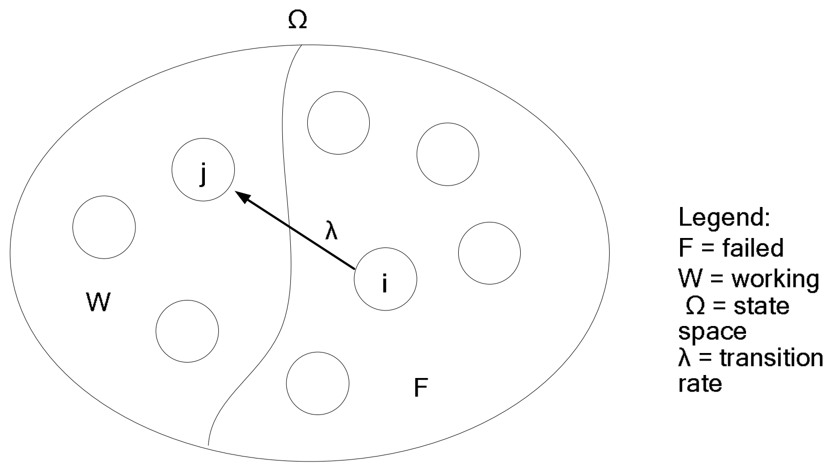


Fig. 3.2. Partition of the state space into working (W) and failed (F) domains.

The probability of failure, P_F, is the probability of the combined state F and can be computed by

$$P_F = \sum_{i \in F} p_i$$

The frequency of failure, f_F, is the sum of the failure state probabilities each multiplied by the total rate of transitions from the respective failure state to the success domain, i.e.,

$$f_F = \sum_{i \in F} \left(p_i \sum_{j \in W} \lambda_{ij} \right)$$

The mean system failure duration, T_F, is the mean duration of stays in the failure domain,

$$T_F = \frac{P_F}{f_F} = \frac{\sum_{i \in F} p_i}{\sum_{i \in F} \left(p_i \sum_{j \in W} \lambda_{ij} \right)}$$

Power System Reliability Example

Using a two-state model for each component and considering only N-1 and N-1-1 line and bus outages, compute the frequency, duration and probability of failure for each load in the system shown in Figure E1. The parallel lines are fully redundant and not on the same structure (there are no common mode failures).

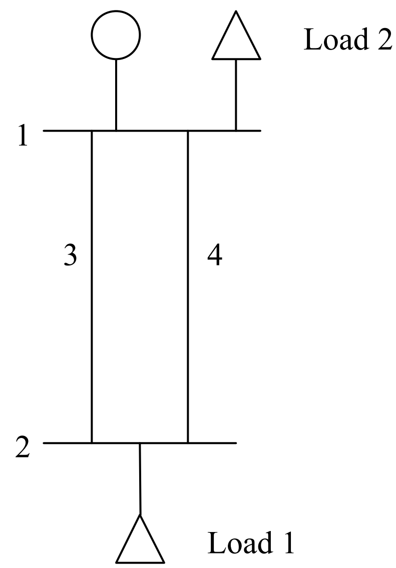


Figure E1. Example System

The component reliability data is as follows

Buses

Component	Type	λ (1/yr)	r (min)	μ (1/yr)
1	1	0.05	200	2628
2	2	0.04	200	2628

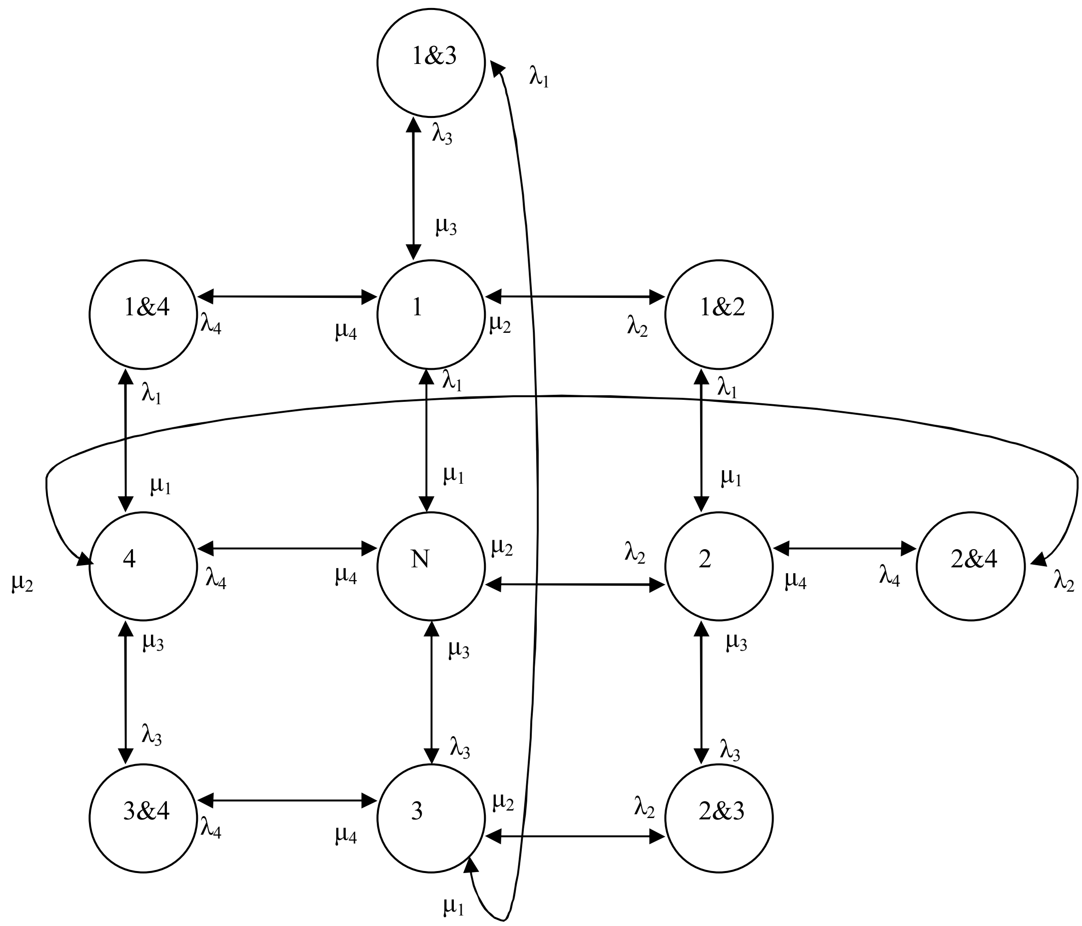
Lines

Component	From	To	λ (1/yr)	r (min)	μ (1/yr)
3	1	2	0.5	200	2628
4	1	2	0.8	300	1752

Where, λ is the failure rate, r is the mean time to repair and μ is the repair rate, $\mu = 1 / r$. Note that all transition rates must have the same units.

Solution

1) Define the state space.



N = Normal Condition, N-0 (all components in service)
1 = component 1 out of service
1 & 3 = component 1 and component 3 out of service

2) Build the state transition matrix.

	N	1	2	3	4	1 & 2	1 & 3	1 & 4	2 & 3	2 & 4	3 & 4
N	$-(\lambda1+\lambda2+\lambda3+\lambda4)$	$\lambda1$	$\lambda2$	$\lambda3$	$\lambda4$						
1	$\mu1$	$-(\mu1+\lambda2+\lambda3+\lambda4)$				$\lambda2$	$\lambda3$	$\lambda4$			
2	$\mu2$		$-(\mu2+\lambda1+\lambda3+\lambda4)$			$\lambda1$			$\lambda3$	$\lambda4$	
3	$\mu3$			$-(\mu3+\lambda1+\lambda2+\lambda4)$			$\lambda1$		$\lambda2$		$\lambda4$
4	$\mu4$				$-(\mu4+\lambda1+\lambda2+\lambda3)$			$\lambda1$		$\lambda2$	$\lambda3$
1 & 2		$\mu2$	$\mu1$			$-(\mu2+\mu1)$					
1 & 3		$\mu3$		$\mu1$			$-(\mu3+\mu1)$				
1 & 4		$\mu4$			$\mu1$			$-(\mu4+\mu1)$			
2 & 3			$\mu3$	$\mu2$					$-(\mu3+\mu2)$		
2 & 4			$\mu4$		$\mu2$					$-(\mu4+\mu2)$	
3 & 4				$\mu4$	$\mu3$						$-(\mu4+\mu3)$

3) Solve for the steady-state probabilities.

Dense LU factorization was used to solve for the steady-state probabilities. Pivoting was not needed since the state transition matrix is diagonally dominant. The code is shown below. (Note the transpose could be avoided by creating $\mathbf{Q}^T$ instead of **Q** at the start.)

```
For i = 0 To n - 1          'transpose
    For j = i + 1 To n
        temp = Q(i, j)
        Q(i, j) = Q(j, i)
        Q(j, i) = temp
    Next j
Next i

For j = 0 To n              'normalize
    Q(n, j) = 1
Next j
p(n) = 1

For j = 0 To n              'LU factorization
    qjj = Q(j, j)
    For k = j To n
        qkj = Q(k, j)
        For i = 0 To j - 1
            qkj = qkj - Q(k, i) * Q(i, j)
        Next i
        Q(k, j) = qkj
    Next k
    qjj = Q(j, j)
    For k = j + 1 To n
        qjk = Q(j, k)
        For i = 0 To j - 1
            qjk = qjk - Q(j, i) * Q(i, k)
        Next i
        Q(j, k) = qjk / qjj
    Next k
Next j

For k = 0 To n              'forward sub
    qkk = Q(k, k)
    yk = p(k)
    For j = 0 To k - 1
        yk = yk - Q(k, j) * p(j)
    Next j
    p(k) = yk / qkk
Next k

For k = n To 0 Step -1      'back sub
    yk = p(k)
    For j = k + 1 To n
        yk = yk - Q(k, j) * p(j)
    Next j
    p(k) = yk
Next k
```

The steady-state probabilities are

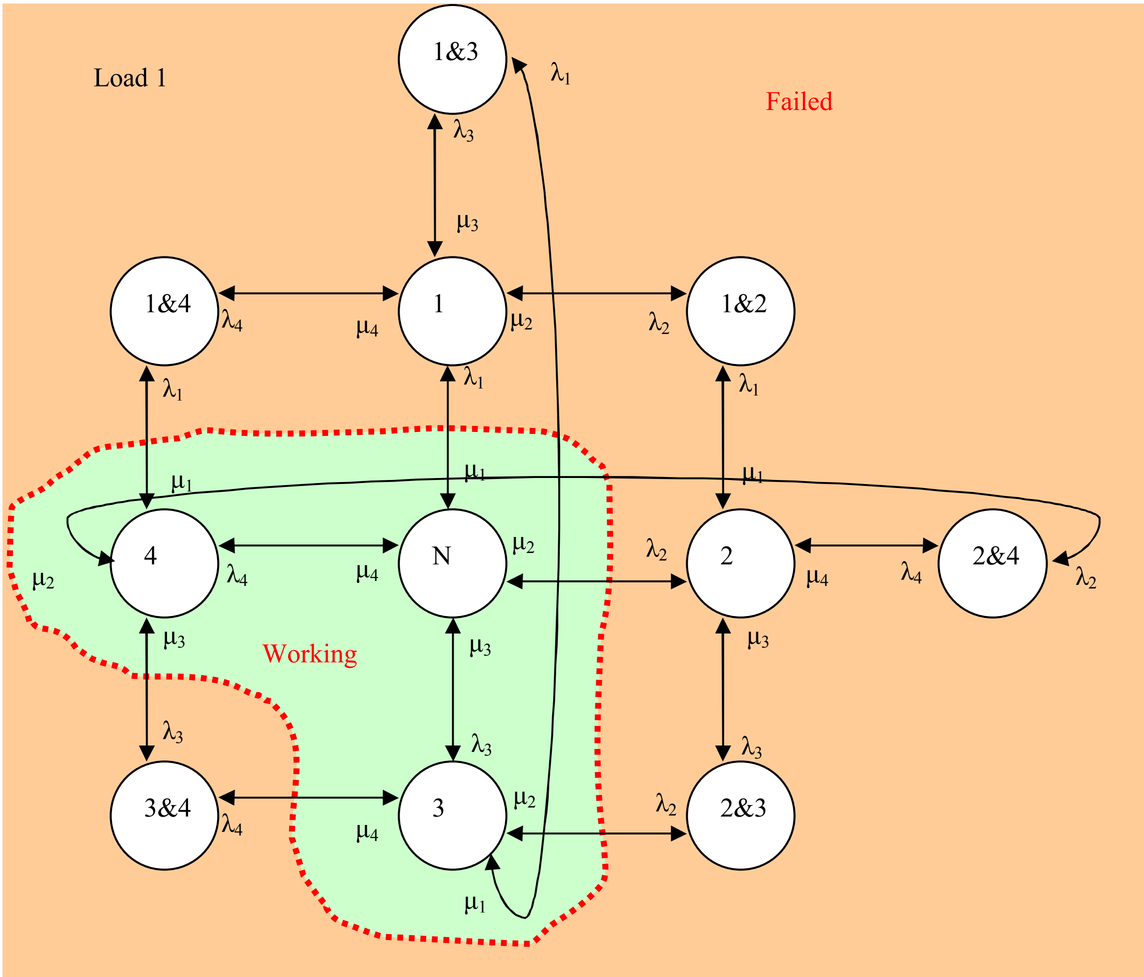
State	Steady-State Probability
N	0.999319228
1	1.90129E-05
2	1.52103E-05
3	0.000190129
4	0.00045631
1 & 2	2.8939E-10
1 & 3	3.61737E-09
1 & 4	8.6817E-09
2 & 3	2.8939E-09
2 & 4	6.94536E-09
3 & 4	8.6817E-08

4) Partition the state space by determining the cut sets for each load point.

State	Load 1	Load 2
N		
1	X	X
2	X	
3		
4		
1 & 2	X	X
1 & 3	X	X
1 & 4	X	X
2 & 3	X	
2 & 4	X	
3 & 4	X	

X = Failure

5) Compute probability, frequency and duration of failure for each load point.

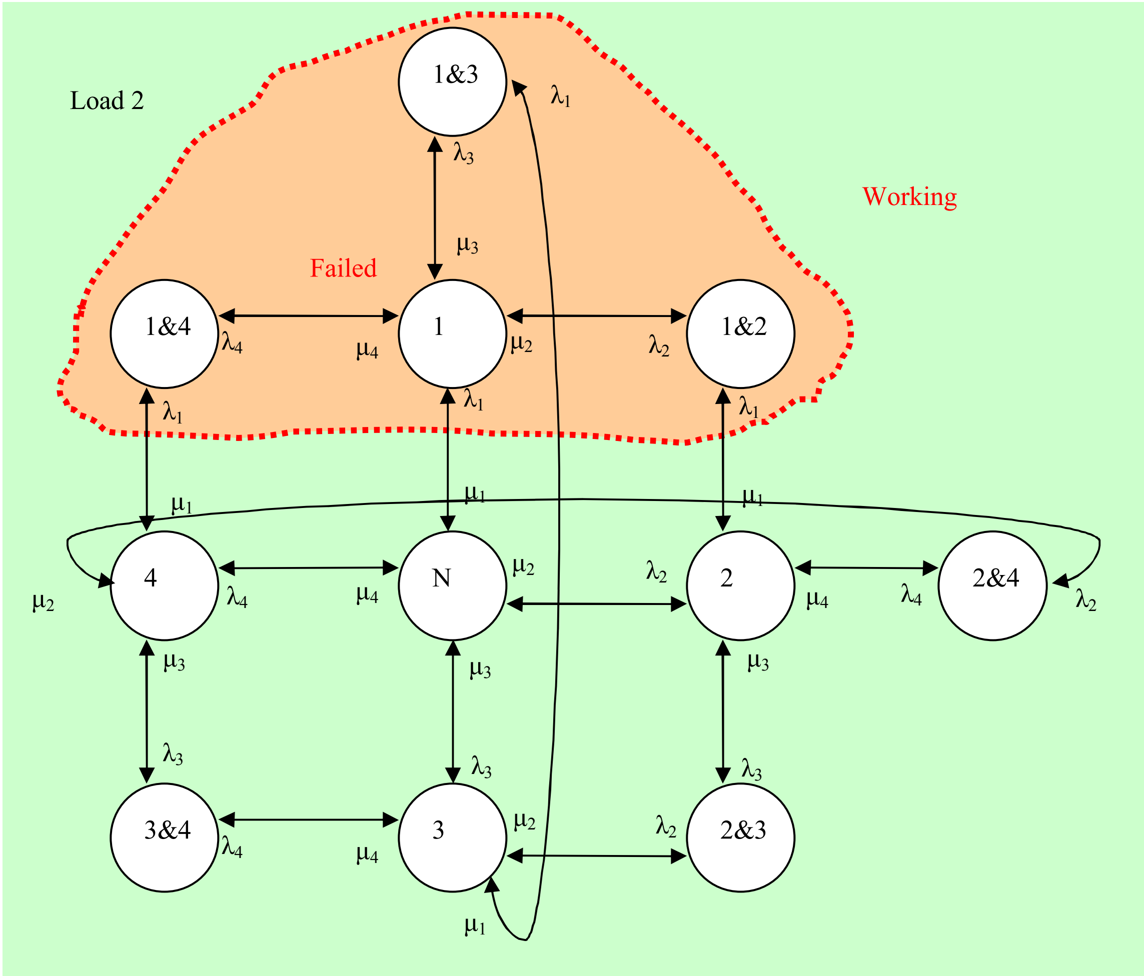


For Load 1

$p_F = 1 - p_W = 1 - p_N - p_3 - p_4 = 3.43E\text{-}05 = 18.05$ minutes/yr

$f_F = p_1 u_1 + p_2 u_2 + p_{1\&4} u_1 + p_{2\&4} u_2 + p_{3\&4}(u_3 + u_4) + p_{1\&3} u_1 + p_{2\&3} u_2 = 0.0904$ failures/yr

$T_F = p_F / f_F = 0.00038$ yrs $= 199.67$ minutes



For Load 2

$p_F = p_1 + p_{1\&2} + p_{1\&3} + p_{1\&4} = 1.903E\text{-}05 = 10$ minutes/yr

$f_F = p_1u_1 + p_{1\&2}u_1 + p_{1\&3}u_1 + p_{1\&4}u_1 = 0.05$ failures/yr

$T_F = p_F/f_F = 0.00038$ yrs = 200 minutes