2.1 Introduction to Linear Systems

A line in the xy-plane can be represented by an equation of the form : $a_1x + a_2y = b$.
This equation is said to be linear in the variables x and y. For example, $x + 3y = 6$.
(Note if $x = 0$ then $3y = 6$ so $y = 2$. Likewise $y = 0$ when $x = 6$. Thus the line passes through the points $(0, 2)$ and $(6, 0)$.)
In general we have the definition:

Definition: A **linear equation** in n variables, $x_1, x_2, \ldots, x_n$, has the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n, b$ are real scalars (i.e. constants). The variables $x_1, x_2, \ldots, x_n$ can also be called the unknowns.

Example: $x + 2y + 3z = 4$ (or $x_1 + 2x_2 + 3x_3 = 4$) is a linear equation representing a plane. Points in the plane can be found by taking particular values for two of the variables and using the equation to solve for the third. For instance, if $x = 0$ and $y = 0$, then $3z = 4$ or $z = 4/3$; similarly, if $x = 0$ and $z = 0$, then $y = 2$; finally, if $y = 0$ and $z = 0$, then $x = 4$. Therefore, $x + 2y + 3z = 4$ is the equation of the plane (in 3-d space) which passes through the points $(0, 0, 4/3)$, $(0, 2, 0)$ and $(4, 0, 0)$.

Remark: A linear equation does not involve powers of variables or products of variables. Nor do the variables appear as arguments of trigonometric, exponential or logarithmic functions.

Examples: $3x_1 - x_2 = x_3 + 1$, $x + 5y = -2$ and $\frac{1}{2}x_1 + \frac{3}{2}x_2 = 0$ are linear equations. Whereas $\sqrt{x_1} + 3x_2 = 4$, $2x_1^{-1} + \sin x_2 = 0$ and $x_1 + 2x_2 + 3x_1x_2 = 1$ are not.

Definiton: A **solution** of a linear equation is a sequence of n numbers $s_1, s_2, \ldots, s_n$ such that when $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ is substituted into the equation, the equation is satisfied. The set of all solutions is called the **solution set** or the **general solution**.

Example: Find the solution set of $x + 3y = 6$.
Solution: To find solutions, an arbitrary value must be assigned to one of the variables. First assign x an arbitrary value t. Then $x = t$ and $t + 3y = 6$. Thus $3y = 6 - t$ or $y = 2 - t/3$. The solution set is given by $x = t, y = 2 - t/3;\ t \in \mathbb{R}$.
Then giving t a particular value yields a particular solution of the equation: e.g. $t = 0$; $x = 0, y = 2 - 0 = 2$ or $(x, y) = (0, 2)$.
We could also have assigned y an arbitrary parameter: $y = s$, $x = 6 - 3s$; $s \in \mathbb{R}$. In this case, giving s the particular value $s = 2$ yields the same solution as before: $y = s = 2, x = 6-3(2) = 0$ or $(x, y) = (0, 2)$.

Example: Find the solution set of $x + 2y + 3z = 4$.
Solution: To find solutions in this case, arbitrary values must be assigned to any two of the variables and then the equation is solved for the third variable. For instance, if $x = t, y = s$, then $t + 2s + 3z = 4$ giving $3z = 4 - t - 2s$ and $z = 4/3 - t/3 - 2s/3$ ($s, t \in \mathbb{R}$).
Then each pair of values s, t gives a point in the plane, e.g. $t = 2, s = 1$ gives $x = 2, y = 1, z = 4/3 - 2/3 - 2/3 = 0$.

Definition: A **system of linear equations** or **linear system** is a finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$. A **solution** of a linear system is a sequence of n numbers $s_1, s_2, \ldots, s_n$ if $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ is a solution of **every** equation in the system.

Example:

$$\begin{aligned} x + y &= 4 \\ 2x - y &= 5 \end{aligned}$$

This is a system of 2 equations in 2 unknowns (called a 2×2 linear system). $x = 3, y = 1$ is a solution of the system since these values satisfy both equations: $3 + 1 = 4$ and $2(3) - 1 = 5$. $x = 2, y = 2$ is not as it is a solution of the first equation only: $2 + 2 = 4$ but $2(2) - 2 = 2 \neq 5$.

Remark: Not all systems of equations have solutions. For instance, consider the system:

$$\begin{aligned} x + y &= 4 \\ 2x + 2y &= 6 \end{aligned}$$

Multiply the second equation by 1/2:

$$\begin{aligned} x + y &= 4 \\ x + y &= 3 \end{aligned}$$

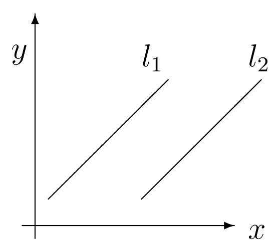
These are contradictory equations and thus there is no solution to the system.

Definition: A system of equations that has no solution is said to be **inconsistent**. If there is at least one solution it is called **consistent**.

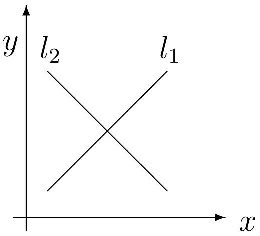
Remark: There are only 3 possibilities: every system of linear equations has either no solutions, exactly one solution or infinitely many solutions.

To see this consider, for instance, a system of 2 equations in the variables x and y (a 2×2 system). Each of the equations represents a line and there are only three possible cases:

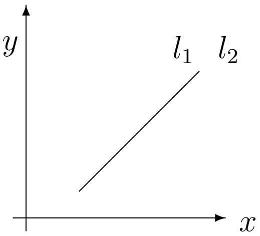
(i)



(ii)



(iii)



(i) The lines are parallel,

(ii) The lines intersect at a single point,

(iii) The lines coincide.

An arbitrary system of m equations in n variables (an $m \times n$ linear system) can be written as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \dots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

The double subscript shows where the coefficient appears in the system: a_{ij} is the coefficient in the ith equation multiplying variable x_j. In order to determine the solutions of a linear system, it is usually most convenient to write the system in matrix form. A **matrix** is a rectangular array of numbers and the **augmented matrix** for a linear system is given by

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & | & b_2 \\ \vdots & \vdots & \vdots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & | & b_m \end{pmatrix}$$

Examples:

System $\begin{array}{ccccc} 2x & + & 3y & = & 7 \\ 4x & - & y & = & -2 \end{array}$ Augmented matrix $\begin{pmatrix} 2 & 3 & | & 7 \\ 4 & -1 & | & -2 \end{pmatrix}$.

System $\begin{array}{ccccccc} x & + & y & - & 5z & = & 14 \\ & & 3y & + & 2z & = & -1 \\ -x & - & 4y & + & z & = & 6 \end{array}$ Augmented matrix $\begin{pmatrix} 1 & 1 & -5 & | & 14 \\ 0 & 3 & 2 & | & -1 \\ -1 & -4 & 1 & | & 6 \end{pmatrix}$.

The basic method behind solving a system of equations is to replace the given system by a new system which has the same solution set but is easier to solve. The new system is obtained by applying a sequence of operations to the original system in order to eliminate the unknowns. The three types of operations allowed are

1. Multiplying an equation by a non-zero constant.
2. Swapping two equations.
3. Adding a multiple of one equation to another equation.

As the rows of the augmented matrix for a system correspond to the equations in the system, these operations correspond to the following operations performed on the matrix:

1. Multiplying a row by a non-zero constant.
2. Swapping two rows.
3. Adding a multiple of one row to another row.

These are called **elementary row operations**.

Example:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & -3 \\ x & + & 3y & + & 6z & = & -13 \\ & & 2y & + & 3z & = & -5 \end{array} \qquad \begin{pmatrix} 1 & 2 & 2 & | & -3 \\ 1 & 3 & 6 & | & -13 \\ 0 & 2 & 3 & | & -5 \end{pmatrix}$$

$$E_2 \longrightarrow E_2 - E_1 \qquad R_2 \longrightarrow R_2 - R_1$$

$$\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & -3 \\ & & y & + & 4z & = & -10 \\ & & 2y & + & 3z & = & -5 \end{array} \qquad \begin{pmatrix} 1 & 2 & 2 & | & -3 \\ 0 & 1 & 4 & | & -10 \\ 0 & 2 & 3 & | & -5 \end{pmatrix}$$

$$E_3 \longrightarrow E_3 - 2E_2 \qquad R_3 \longrightarrow R_3 - 2R_2$$

$$\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & -3 \\ & & y & + & 4z & = & -10 \\ & & & - & 5z & = & 15 \end{array} \qquad \begin{pmatrix} 1 & 2 & 2 & | & -3 \\ 0 & 1 & 4 & | & -10 \\ 0 & 0 & -5 & | & 15 \end{pmatrix}$$

$$E_3 \longrightarrow -\frac{1}{5}E_3 \qquad R_3 \longrightarrow -\frac{1}{5}R_3$$

$$\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & -3 \\ & & y & + & 4z & = & -10 \\ & & & & z & = & -3 \end{array} \qquad \begin{pmatrix} 1 & 2 & 2 & | & -3 \\ 0 & 1 & 4 & | & -10 \\ 0 & 0 & 1 & | & -3 \end{pmatrix}$$

A value for one of the variables, z, has been found and thus the solution of the system can now be obtained by using this to find the values of the other variables: substituting $z = -3$ in E_2 gives $y - 12 = -10$ or $y = 2$, then substituting $y = 2$, $z = -3$ in E_1 gives $x + 4 - 6 = -3$ or $x = -1$.
Alternatively, we could carry on in a similar manner to before to obtain the solution:

$E_2 \longrightarrow E_2 - 4E_3$

$$\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & -3 \\ & & y & & & = & 2 \\ & & & & z & = & -3 \end{array}$$

$R_2 \longrightarrow R_2 - 4R_3$

$$\begin{pmatrix} 1 & 2 & 2 & | & -3 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & -3 \end{pmatrix}$$

$E_1 \longrightarrow E_1 - 2E_3$

$$\begin{array}{rcrcrcr} x & + & 2y & & & = & 3 \\ & & y & & & = & 2 \\ & & & & z & = & -3 \end{array}$$

$R_1 \longrightarrow R_1 - 2R_3$

$$\begin{pmatrix} 1 & 2 & 0 & | & 3 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & -3 \end{pmatrix}$$

$E_1 \longrightarrow E_1 - 2E_2$

$$\begin{array}{rcrcrcr} x & & & & & = & -1 \\ & & y & & & = & 2 \\ & & & & z & = & -3 \end{array}$$

$R_1 \longrightarrow R_1 - 2R_2$

$$\begin{pmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & -3 \end{pmatrix}$$

Solution: $x = -1,\ y = 2,\ z = -3.$