Notes: Superposition

Linearity and Superposition
The properties of Superposition and Homogeneity, like Symmetry, are commonly used intuitively in many engineering circumstances. Their use often simplifies understanding of otherwise complex affairs. There is, however, a well-defined meaning to these properties. The following brief note provides some background.

Formally an equation is said to be linear if it satisfies the conditions of superposition and scaling. Using a linear function of one variable for illustration:

$F(a) + F(b) = F(a+b)$	Superposition (Additive) Property
$kF(a) = F(ka)$	Scaling (Homogeneity) Property

Note that a linear equation does **not** necessarily have the property of linearity as defined here. For example the equation $y = 2x + 3$ does not have the property of superposition, i.e., $y(x1+x2) \neq y(x1 + (x2)$. Nor does it have the scaling property, i.e., $2y(x) \neq y(2x)$. However the special case of a line through the origin does have the linearity property.

The volt-ampere relations of circuit elements such as resistors are linear, i.e., have both the requisite superposition and scaling properties. Moreover KCL and KVL are linear in the branch currents and branch voltages respectively. And in solving for the branch currents and voltages of a network the algebraic operations used, addition or multiplication by a constant, do not affect the linearity property of the equations. Hence it follows that circuit equations, expressions for voltages and currents of networks comprised of linear circuit elements, have the linearity property. The knowledge that a particular network has the linearity property is very useful because it enables us to apply superposition and scaling to draw important conclusions without actually having to solve the circuit. It is worth emphasizing again that not all circuits have the linearity property; in particular the analysis of nonlinear circuits is more involved than that for linear circuits.

Suppose a certain linear circuit contains two voltage sources E1 and E2. It follows then that any branch voltage V_j (or current I_j) can be related to the sources by a linear equation of the form $V_j = aE1 + bE2$, where a and b are constants (different generally for different branches, of course). One implication of this expression is that one can calculate the voltage produced by one source with the other 'turned off', and vice versa, and then add the two results to obtain the branch voltage when both sources are active. In mathematical form this is $V_j(E1, E2) = V_j(E1, 0) + V_j(0, E2)$.

Another implication: if both source strengths are increased by a multiplicative factor m then all branch currents and voltages increase by the same factor, i.e., $V_j(mE1, mE2) = mV_j(E1, E2)$. Actually it is an everyday occurrence for this sort of scaling to be applied without thinking about the formalities involved. For example one intuitively 'turns up' a power supply voltage to increase a branch voltage or current to a desired level.

Consider the following application of linearity. Suppose all the resistance values in a circuit having only voltage sources are increased by a factor m. It follows that all branch voltages remain unchanged, but all branch currents decrease by a factor m. Why? The voltages do not scale because the source voltages are not changed, and scaling (multiplication by a constant) would mean KVL could not be satisfied. Ohm's Law indicates the resistive branch currents do scale; scaling all the currents by the same factor means KCL remains satisfied.

On the other hand if the circuit contained only current sources then the branch currents do not change and KCL is preserved. But the resistive branch voltages increase by a factor m thus satisfying Ohm's Law and KVL.

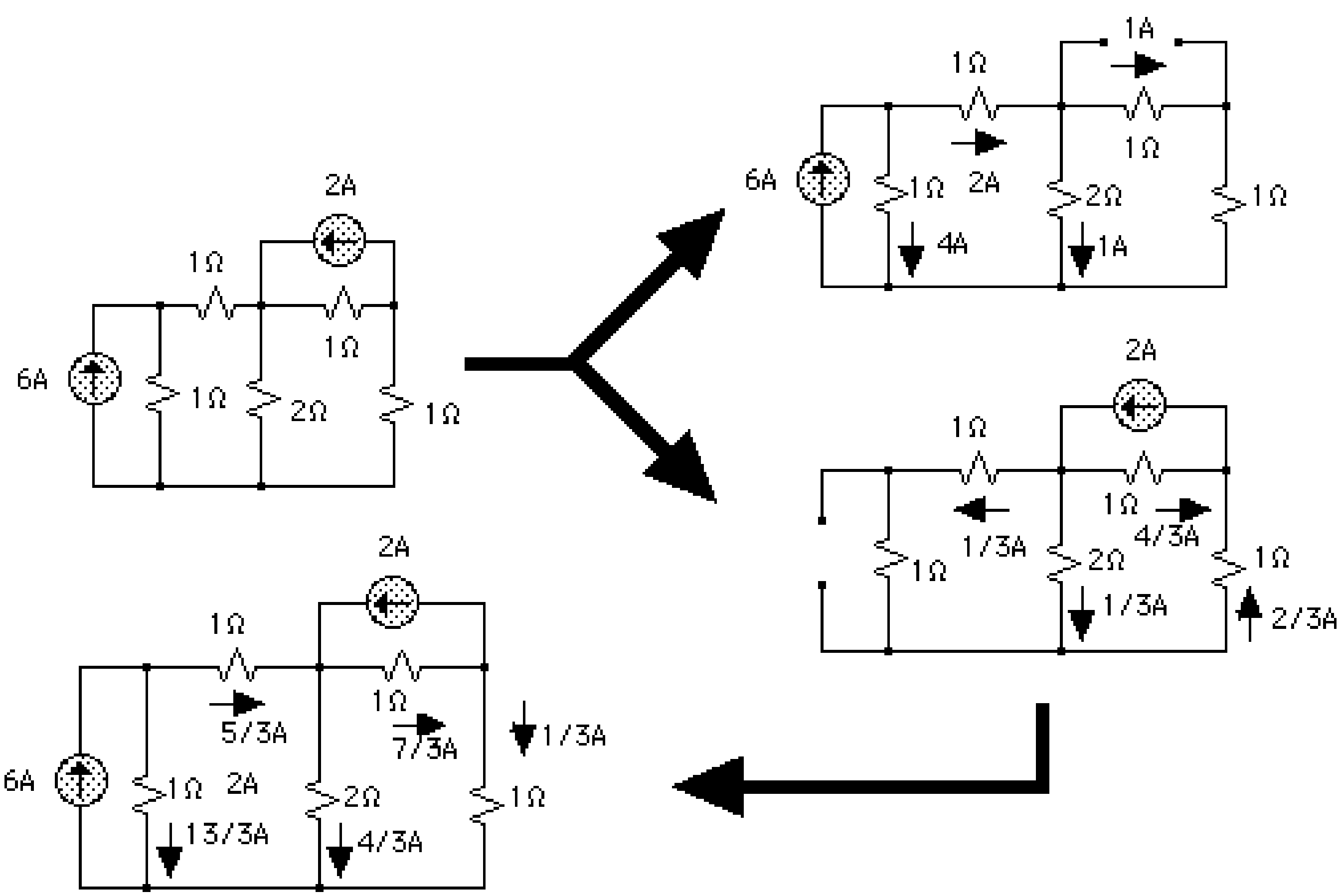
For a circuit containing both voltage and current sources apply Superposition; analyze the circuit in two steps, first with one type of source turned off and then with the other type turned off.

One more illustration, one out of the many uses of linearity. Superposition tells us that $V_j(E1, E2) = V_j(E1) + V_j(E2)$. We can solve separately for the branch currents and voltages produced by E1 when E2 is removed (made zero), and then for the branch currents and voltages produced by E2 when E1 is removed. Now E1 and E2 can be scaled separately and superimposed to determine readily the branch voltage obtained for varying proportions of the two source strengths, i.e.,

$$V_j(aE1, bE2) = V_j(aE1, 0) + V_j(0, bE2) = aV_j(E1, 0) + bV_j(0, E2)$$

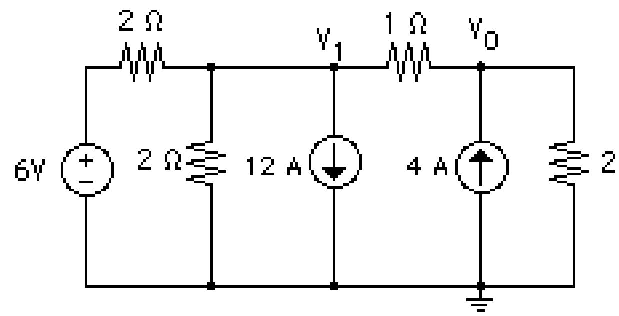
Old Lecture Problem

The diagram below illustrates the superposition example worked out in class. The circuit to be analyzed is that at the top, left. The bifurcating arrows indicate the circuits with respectively, the 2A source turned off (strength -> 0) and the 6A source turned off (strength -> 0). Each of the circuits then is analyzed to determine (here) just the branch currents. The method of analysis is not material here. There is only one solution. (Simplifying series-parallel resistance reductions can be used, but illustration of superposition is the object here.)



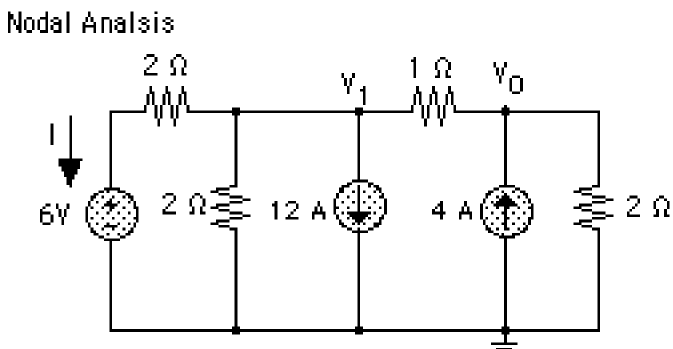
Finally the two solutions are combined (superimposed) to obtain the solution when <u>both</u> sources are active. Verify that KVL, KCL, and the V-A relations are satisfied.

On Analyzing a Circuit



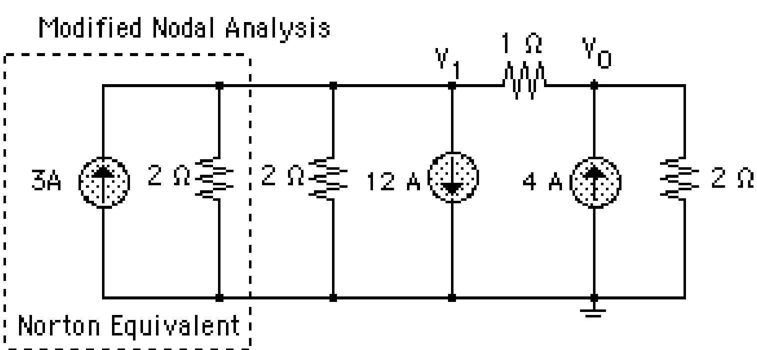
Consider the circuit drawn to the right. The basic circuit question, leaving aside a consideration of the effectiveness of the circuit in performing whatever function it was designed to perform, is to determine the several branch voltages and currents. It is a secondary but nevertheless important consideration to decide the method used to perform the analysis. Fundamentally it is necessary and sufficient to use KVL, KCL, and the V-A relations to perform the analysis; the solution is unique. Various procedures have been described in which these relations are used in different ways to perform an analysis. As does any skilled worker choose from among the several variations whichever procedure seems convenient. (Incidentally quite often more time and effort can be spent trying to decide the most efficient procedure than would be required to find a solution inefficiently.) Below the circuit is analyzed again and again and again.

Nodal Analysis For example the diagram provides at least a subliminal suggestion to use a nodal analysis. Of course there is the matter of the voltage source; its V-A relation does not specify a relation between the source voltage and current. No matter. Simply introduce the source current as a new unknown, and write the KCL equations at the node using this new variable. Of course adding a new variable means we need another independent equation, and this is obtained directly: $V_1 = 2I + 6$. The node-to-datum matrix expression is shown to the right of the diagram below; solve (for example) for $V_1 = -4.75V$.



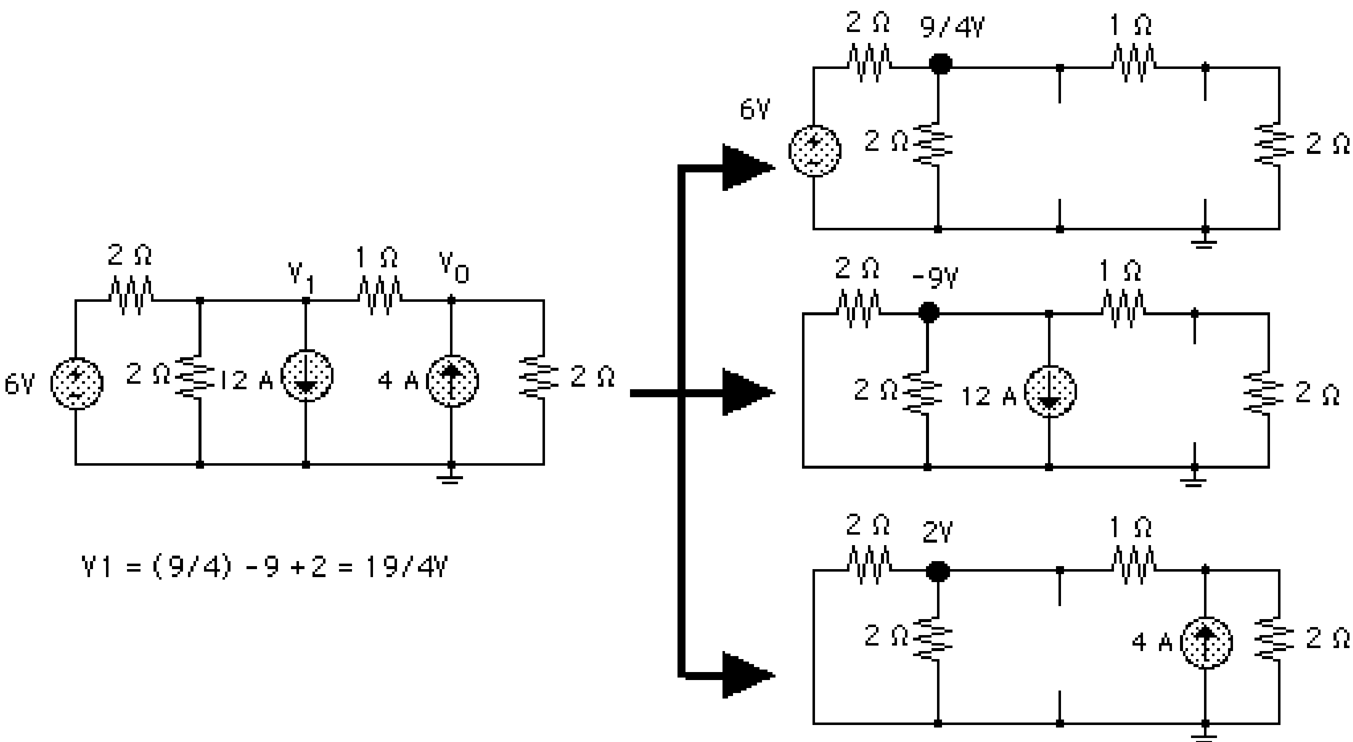
$$\begin{vmatrix} -12 \\ 4 \\ 6 \end{vmatrix} = \begin{vmatrix} 1+1/2 & -1 & 1 \\ -1 & 1+1/2 & 0 \\ 1 & 0 & 2 \end{vmatrix} \begin{vmatrix} V_1 \\ V_0 \\ I \end{vmatrix}$$

Modified Nodal Analysis The diagram below shows the series connection of the 6V source and the 2Ω resistor replaced by a Norton equivalent circuit. Why a Norton replacement? Because it will introduce a current source in parallel with a resistor, simplifying a nodal analysis. Of course again $V_1 = -4.75V$; the solution is unique.

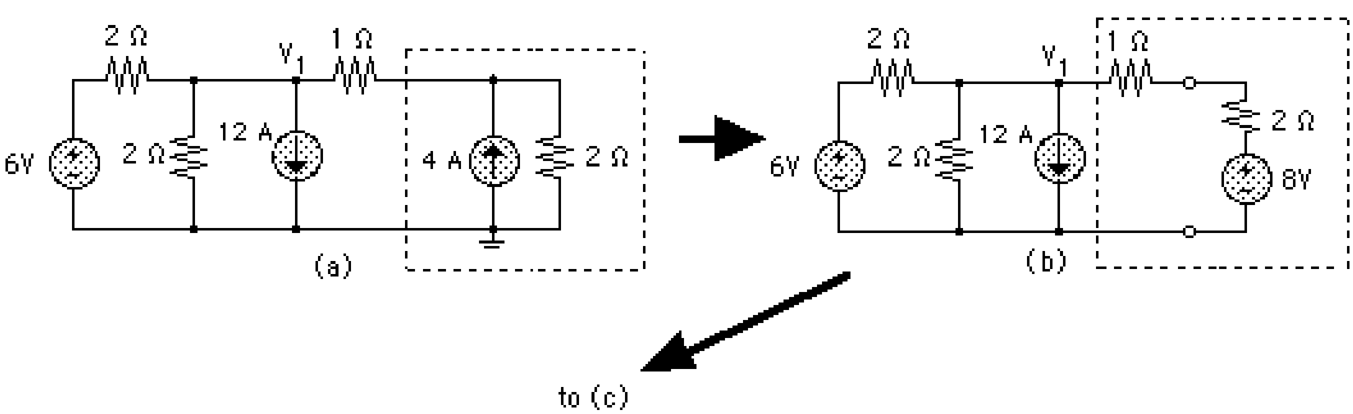


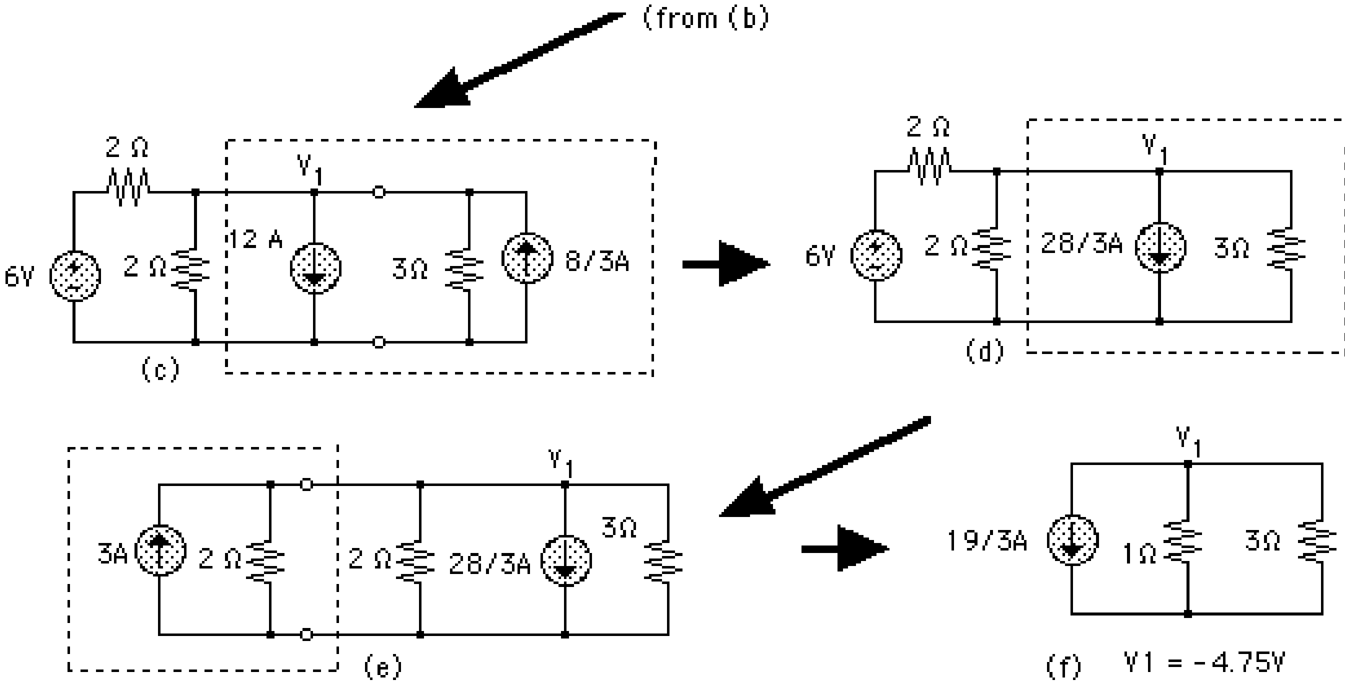
$$\begin{vmatrix} -9 \\ 4 \end{vmatrix} = \begin{vmatrix} 1+1/2+1/2 & -1 \\ -1 & 1+1/2 \end{vmatrix} \begin{vmatrix} V_1 \\ V_0 \end{vmatrix}$$

Superposition The diagram below illustrates the application of superposition to the process of analyzing the circuit. Although there are three circuits to analyze in obtaining the composite result each circuit separately is easily analyzed.

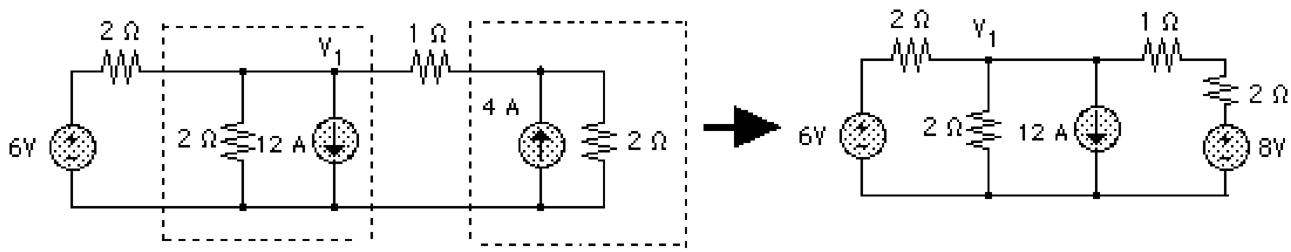


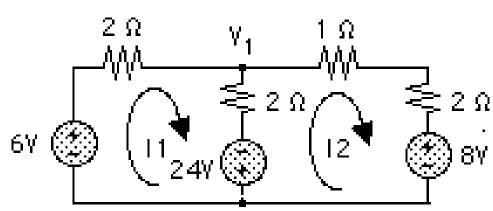
Iterated Thevenin-Norton Transformations There are an almost endless variety of equivalent circuit transformations that can be applied if desired (and possible). In the diagrams following a series of Thevenin/Norton conversions is made. The 4A-2Ω combination of elements (dashed box) in circuit (a) is replaced in (b) by a Thevenin equivalent circuit. The 1Ω-2Ω series combination of resistors is replaced in (c) by a single 3Ω equivalent resistor, and the conversion to a Norton equivalent then made. Note that the two parallel current sources are combined into a single equivalent source in (d). In (e) a Thevenin transformation is made (as described before). Combine the sources as in (f) and compute V_1 directly.





Mesh Current Analysis A mesh current analysis would appear to be awkward because the constitutive relation for a current source does not declare an associated voltage. One could proceed by declaring two new voltage variables and adding two new independent equations. However two Norton to Thevenin transformations change the circuit as shown, and the analysis proceeds in the usual fashion.

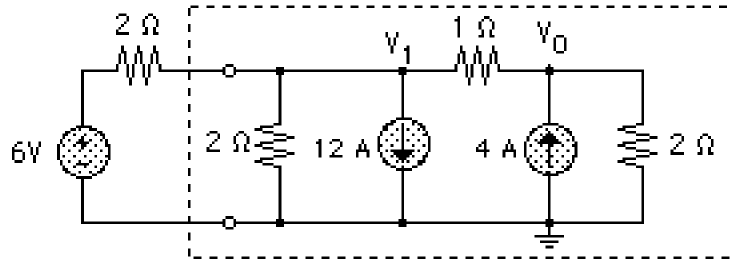


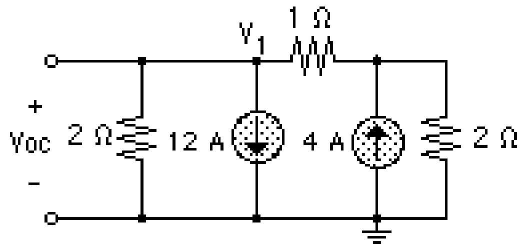


$$\begin{vmatrix} 6+24 \\ -24-8 \end{vmatrix} = \begin{vmatrix} 2+2 & -2 \\ -2 & 2+1+2 \end{vmatrix} \begin{vmatrix} I1 \\ I2 \end{vmatrix}$$

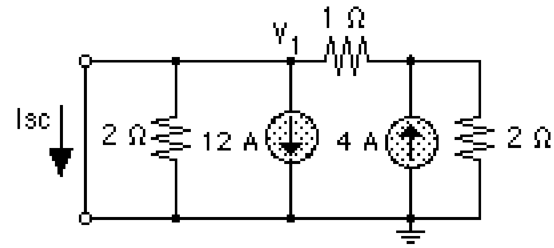
$$V1 = 6 - 2I1 = -4.75V$$

Go-For-Broke Thevenin Transformation Look 'into' the indicated terminals and replace the circuit to the right by its Thevenin equivalent as indicated.

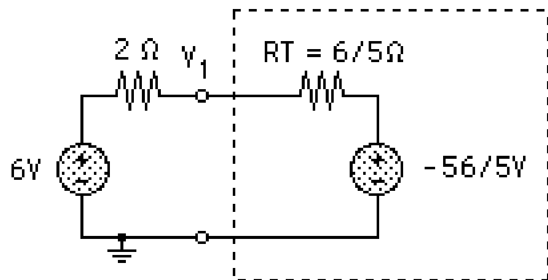




Voc = -56/5V



Isc = -28/3A



Solve for V1 = -4.75V