Chapter 2

- Matrix Operations -

Introduction

We shall present here a brief introduction to matrices and the MATLAB commands relevant to working with them.

Basic Understanding

Matrix algebra approach is a useful means of compactly presenting lengthy and sometimes complex formulas. A matrix is a rectangular array of elements arranged in horizontal rows and vertical columns. The number of rows and columns may vary from one matrix to another, so we conveniently describe the size of a matrix by giving its dimension, that's the number of its rows and columns. A matrix is said to have size n x m, read "n by m" if it has n rows (horizontal lines) and m columns (vertical lines). The number of rows is always stated first. A square matrix is one for which m=n. The most direct way to create a matrix in MATLAB is to type the entries in the matrix between square brackets one row at a time.

How to enter a matrix?

You can enter numerical matrices in a number of ways. Typically, we will be entering data either manually or reading it from a spreadsheet. To enter a matrix, use commas on the same row, and semicolons to separate columns.

Purpose	Way 1	Way 2
Separate the entries in the same row	Type a comma (,)	Press the "Space Bar"
Indicate the beginning of a new row	Type a semicolon (;)	Type a newline

Example

– How to enter a matrix-

For example, the matrix A which is mathematically defined by

$A = \begin{bmatrix} 1 & 2 & 9 \\ 4 & 7 & 5 \\ 3 & 1 & 6 \end{bmatrix}$ is described in MATLAB by

$>> A = \left[\underbrace{1\ 2\ 9}_{first\ row}; \ \underbrace{4\ 7\ 5}_{second\ row}; \ \underbrace{3\ 1\ 6}_{third\ row} \right]$ % create a 3-by-3 square matrix which is named A

This results in a 3x3 matrix, which looks like

```
A =
     1     2     9
     4     7     5
     3     1     6
```

MATLAB displays matrices without braces

Building Large matrices

Assembling Matrices

Large matrices can be assembled from smaller matrix blocks:

Example

- Adding a row to an existing matrix-

```
>>A=[1 2 9; 4 7 5; 3 1 6];
>>B=[A; 11 12 13]                    % add one row to matrix A
 B =
      1     2     9
      4     7     5
      3     1     6
     11    12    13
```

(Continue on the next page)

```
>>C=[A; A; A]

 C =
      1     2     9
      4     7     5
      3     1     6
      1     2     9
      4     7     5
      3     1     6
      1     2     9
      4     7     5
      3     1     6
```

Size Command

We can determine the size of a vector or matrix by using the **size** command.

```
>>size(A)          % return the size of A
>>size(A,1)        % return the number of rows in A
>>size(A,2)        % return the number of columns in A
```

Individual Elements

Individual elements of a matrix can be referenced via indices enclosed within parentheses. The first index refers to the row number, and the second index refers to the column number.

Example

- Reference Individual Elements-

```
>>A(2,1)          % reference the second element of the first row

results in

ans =
      4
```

Reassigning Matrices Values & Colon Operator

A. Reassigning Matrices Values

It is possible, under MATLAB, to alter the elements by reassigning their values.

```
>>A(2,1)=9                % change the second element in the first column to 9
```

B. Colon Operator

The colon character (:) means several different things in MATLAB.

In round brackets, a colon means everything in a row or column and is typically used to extract data from a matrix.

>>A(:,3) means everything in column 3 of A.
>>A(:) rearranges everything in A into one long column vector

Practice

- Reassigning Matrices Values & Colon Operator-

$$A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 9 & 2\end{bmatrix}, B=\begin{bmatrix}1\\ 5\\ 9\end{bmatrix}, C=\begin{bmatrix}2 & 1 & 0\end{bmatrix}$$

```
>>A=[ 1 2 3; 4 5 6; 7 9 2];    % specify matrix A
>>B=[1; 5; 9];                % specify column vector B
>>C=[2 1 0];                  % specify the row vector C
>>A(:,1)=B;                   % replace the first column of A by B
>>A(2,:)=C;                   % replace the second row of A by C
```

Additional Commands for Building Large Matrices

Furthermore, the following commands are quite handy:

```
>>A(:,j);                 % correspond to the jth-column of A
>>A(i,:);                 % correspond to the ith-row of A
>>A(:,k:m);               % correspond to [A(:,k) A(:,k+1),…A(:,m)]
>>A(:,2)=[ ];             % delete the second column of A
```

Vectors can be viewed as special cases of matrices. Therefore, vectors are generated the same may as matrices.

Practice

- Additional Commands for Building Large Matrices- (1)

```
>>V=[3 5 7]   <enter>              % define a row vector A

V =
     3       5       7
```

There is another useful operator for working with matrices, this is the word **end** which indicates the highest value (dimension) of the matrix.

Practice

- Additional Commands for Building Large Matrices- (2)

```
>>A=[1 2 3; 4 5 6; 7 8 9];
>>d=A(1:2,end)

d =
     3
     6
```

It returns the values 3 and 6. That is, the first two values in the 3rd column (the end column of the matrix)

Special Matrices

Four Kinds of Special Matrices

A **diagonal matrix** is a matrix where only the diagonal entries are non-zero. An **identity matrix**, **I**, is the diagonal matrix with diagonal consisting of all 1's. An **upper triangular matrix** is a matrix whose entries lying below the diagonal are all zero. **A lower triangular matrix** is a matrix whose entries lying above the diagonal are all zero.

A. Diagonal Matrix

The diagonal matrix A is one whose elements off the main diagonal are all equal to zero, while those along the main diagonal are non-zero.

The command **diag** will generate a diagonal matrix with the specified elements on the main diagonal.

Practice

-Special Matrices: The "diag" Command for Diagonal Matrix-

```
>>A=diag([1 2 3])                % generate a diagonal matrix
```

B. Identity Matrix

If A is any matrix, the identity matrix for multiplication is a matrix **I** which satisfies the following relation.

AI = A and IA = A

This matrix, called the identity matrix, is the square matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

That is, all elements in the main diagonal of the matrix, running from top left to bottom right, are equal to 1, all other elements equal zero. Note that the identity matrix is always indicated by the symbol I.

Commands for Special Matrices

MATLAB has several build-in matrices. The command **eye(n)** produces a n-by-n identity matrix. The **zero(n,m)** and **ones(n,m)** command will generates an n-by m matrices willed with zeros, and filled with ones, respectively. **The rand(n)** command will generate an n-by-n matrix whose elements are pseudo-random numbers uniformly

distributed between 0 and 1, while **rand(n,m)** will create a n-by-m matrix with randomly generated entries distributed uniformly between 0 and 1. The **magic(n)** command generate a n-by-n square matrix whose entries constitute a magic square; i.e., the sum of elements along each row, column, or principal diagonal is the same value.

Practice

- Commands for Special Matrices-

(1)

```
>>D=eye(3)   <enter>              % create a 3-by-3 identity matrix

D =
     1     0     0
     0     1     0
     0     0     1

>>H=zeros(2,3)   <enter>          % create a 2-by-3 null matrix

H =
     0     0     0
     0     0     0

>>W=ones(2,3)   <enter>           % create a 2-by-3 matrix with entries equal to 1

W =
     1     1     1
     1     1     1
```

Practice

- Commands for Special Matrices-

(2)

You can allocate memory for 1-D arrays (vectors) using the **zeros** command. The following command allocates a 200-dimensional array:

```
>>y=zeros(200,1)
```

Similarly, you can allocate memory for 2-D matrices. The command

```
>>y=zero(5,6)
```

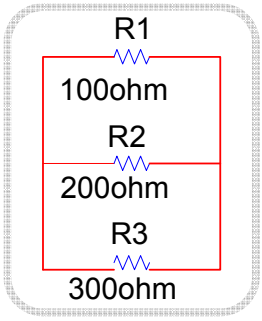
Defines a 5-by-6 matrix

Practice

- Commands for Special Matrices-

(3)

White a MATLAB code to compute the equivalent resistance of three resistors connected in parallel.




```
>> % R1=100, R2=200, and R3=300
>>R=[100 200 300];                % specify vector of resistances
>>R_inv=ones(size(R)./R;          % evaluate 1/R
>>R_eq=1/sum(R_inv);              % evaluate the equivalent resistance
>>R_eq <enter>                    % print the equivalent resistance

R_eq =
      54.5455
```

The Empty Matrix

The empty matrix is represented in MATLAB as [ ]. This is a matrix with dimension zero-by-zero. Therefore, the statement

```
>>B= [ ]
```

enters a zero-by-zero matrix B into the workspace. The empty matrix has some special uses. It can serve, for example, to reduce the size of an existing matrix. For example

```
>>A([1 3],:)=[ ]          % deletes rows one and three from A
>>A=[1 2 3; 4 5 6; 7 8 9];
>>flipud(A)               %flip up-down
>>fliplr(A)               % flip left-right
>>rot90(A,2)              % 2 rotations of 90 degrees (ccw)
>>triu
>>tril
```

Matrix Operations

Addition of Matrices

Matrix addition can be accomplished only if the matrices to be added have the same dimensions for rows and columns. If A and B are two matrices of the same size, then the sum A+B is the matrix obtained by adding the corresponding entries in A and B.

Practice

- Matrix Operations: Addition of Matrices-

```
>>A=[3 4; 5 6];           % define matrix A
>>B=[1 2; 9 7];           % define matrix B
>>C=A+B   <enter>         % compute the sum

C =
     4     6
    14    13
```

If B is any matrix, then the negative of B, denoted by –B, is the matrix whose entries

are the negatives of the corresponding entries in B.

□□***NOTE***□□

A scalar may be added to a matrix of any dimension. If A is a matrix, the expression A+4 is evaluated by adding 4 to each element of A.

Difference of Matrices

If A and B are matrices with the same size, then the difference between A and B denoted A-B is the matrix defined by A-B=A+(-B).

Practice

- Matrix Operations: Difference of Matrices-

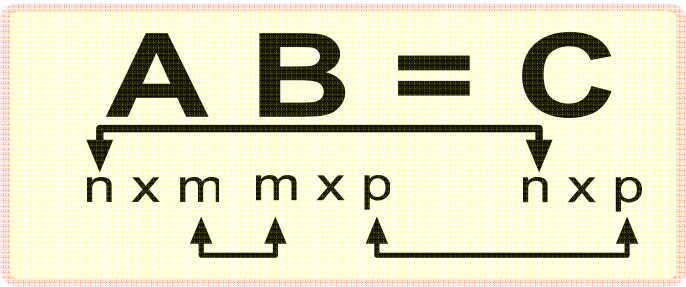
```
>>A=[3 4; 5 6];                % define matrix A
>>B=[1 2; 9 7];                % define matrix B
>>D=A-B   <enter>              % compute the difference

D =
     2     2
    -4    -1
```

Product of Matrices

We can form the product C=A x B only if the number of rows of B, the right matrix, is equal to the number of column of A, the left matrix. Such matrices are said to be conformable. Given an m-by-n matrix A and a k-by-p matrix B, then A and B are conformable if and only if n=k. An element in the ith row and jth column of the product, AB, is obtained by multiplying the ith row of A by the jth column of B.

A straightforward way of checking for conformity is to write the dimensions underneath the matrices as illustrated below:



Observe that the inner two numbers, giving the number of elements in a row of A and column of B, respectively, must be equal. The outer two numbers, indicating the number of rows and A and columns of B, give the dimensions of the product matrix C. Remember that the order of multiplication is important when dealing with matrices.

Practice

-Matrix Operations: Product of Matrices

For the given matrices, obtain the product C=A*B

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix},\quad B=\begin{bmatrix}-2 & 3\\ 4 & 1\end{bmatrix}$$

```
>>A=[1 2; 3 4; 5 6];          % define matrix A
>>B=[-2 3; 4 1];              % define matrix B
>>C= A*B   <enter>            % compute the product matrix C

C =
     6     5
    10    13
    14    21
```

□□***NOTE***□□

Matrix multiplication is not commutative in general, i.e., A*B≠B*A

Multiplication by a Scalar

If A is a matrix and k is a scalar, then the product k* A is defined to be the matrix obtained by multiplying each entry of A by the constant k.

Practice

- Matrix Operations: Multiplication by a Scalar-

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix},\quad B=\begin{bmatrix}7 & 0\\ 5 & 9\end{bmatrix}$$

```
>>A=[1 2; 3 4];          % define matrix A
>>B=[7 0; 5 9];          % define matrix B
>>C=2*A                  % scale A by 2
>>D=3*A+2*B              % scale A by 3, B by 2, and add the results
```

answers

```
C =
     2     4
     6     8
D =
    17     6
    19    30
```

Inner Product

Consider two vectors $X=[x_1, x_2]$, and $Y=[y_1, y_2]$. The inner product (or do product) is defined as $X.Y = x_1 \times y_1 + x_2 \times y_2 = |X||Y|\cos(\theta)$.

Practice

-Matrix Operations: Inner Product-

Find the inner product and the angle θ for the following two vectors:

A=[3 4]
B=[6 7]

```
>>A=[3 4];                % specify vector A
>>B=[6 7];                % specify vector B
>>C=A*B'                  % evaluate the inner product
>>%compute the value of the angle theta
>>theta=acos(C/(sqrt(A*A')*sqrt(B*B')))
```

Some Properties of Matrices

1. A+B=B+A	
2. A+(B+C)=(A+B)+C	
3. A(B+C)=AB+AC	
4. AI=IA=A	% I is the identity matrix
5. 0A=0	% 0 is the null matrix
6. A0=0	
7. A+0=A	

Determinant of a Matrix

Introduction

Associated with any square matrix A is a scalar quantity called the **determent** of the matrix A. A matrix whose determinant is non-zero is called a **non-singular** or **invertible**, otherwise it is called **singular**.

Determinant of a 2-by 2 Matrix

For a 2-by-2 matrix A,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The determinant is equal to $a_{11}a_{22} - a_{21}a_{12}$.

Determinant of a 3-by-3 Matrix

Similarly, for 3-by-3 matrix B,

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant is equal to :

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$$

$$a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{12}\left(a_{21}a_{33} - a_{31}a_{23}\right) + a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right)$$

The command **det** evaluates the determinant of a square matrix.

Practice

-Determinant of a Matrix-

Find the determinant of the square matrix A depicted below.

$$A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 1 & 2\end{bmatrix}$$

```
>>A=[ 1 2 3; 4 5 6; 7 1 2];          % specify matrix A
>>det(A)                             % compute the determinant

ans =
      -21
```

□□***NOTE***□□

For the diagonal matrix B,

$$B=\begin{bmatrix}b_{11} & 0 & 0\\ 0 & b_{22} & 0\\ 0 & 0 & b_{33}\end{bmatrix}$$

The determinant is equal to the product of the diagonal elements, i.e., det(B)= $b_{11}b_{22}b_{33}$.

Rank of a Matrix

The "rank" Command

The rank of a matrix, A, equals the number of linearly independent rows or columns. The rank can be determined by finding the highest-order square sub-matrix that is non-singular. The command rank provides the rank of a given matrix.

Practice

- Rank of a Matrix-

```
>>A=[1 2 3; 4 5 6; 7 8 9];          % define matrix A
>>rank(A)                           % determine the rank of matrix A
```

Inverse of Matrix

If A and B are square matrices such that AB=BA=I, then matrix B is called the **inverse** of A and we usually write it as $B = A^{-1}$. Only a square matrix whose determinant is not zero has an inverse. Such a matrix is called nonsingular. If any row (or column) of a square matrix is some multiple or linear combination of any other row(s) [or column(s)] the matrix will be singular, i.e., have a determinant of zero and have no inverse. The command **inv** provides the inverse of a matrix.

Practice

-Inverse of Matrix: The "inv" Command-

Find the inverse of the matrix A given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 2 \end{bmatrix}$$

```
>>A=[1 2 3; 4 5 6; 7 8 2];          % define matrix A
>>B=inv(A)   <enter>               % compute the inverse of A store result in B
B =
   -1.8095    0.9524   -0.1429
    1.6190   -0.9048    0.2857
   -0.1429    0.2857   -0.1429
```

□□***NOTE***□□

For the diagonal matrix B given by

$$B = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$$

The inverse is also a diagonal matrix C,

$$C = \begin{bmatrix} \frac{1}{b_{11}} & 0 & 0 \\ 0 & \frac{1}{b_{22}} & 0 \\ 0 & 0 & \frac{1}{b_{33}} \end{bmatrix}$$

Quick method for finding the inverse of a 2-by-2 matrix

1. Interchange the elements of the main diagonal
2. Change the signs of element on the secondary diagonal
3. Divide by the determinant

Transpose of a Matrix

The transpose of a matrix is the result of interchanging rows and columns. The transpose is found by using the prime operator (apostrophe), [‘]. In particular, the transpose of a row vector is equal to a column vector and vice versa. For a matrix with complex entries the prime operator yields the “*complex conjugate”* transpose. To obtain a non-conjugate transpose, use the dot-transpose operator “ **.’** ”.

Practice

- Transpose of a Matrix-

Find the transpose of the matrix A

$$A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 2\end{bmatrix}$$

```
>>A=[1 2 3; 4 5 6; 7 8 2];      % specify matrix A
>>B=A'   <enter>               % compute the transpose of A and store result in B

B =
     1     4     7
     2     5     8
     3     6     2
```

Symmetric Matrix

A symmetric matrix can be defined as one which is equal to its transpose. Clearly only matrices can be symmetric. It is easily shown that the sum or difference of two symmetric matrices is also symmetric.

Practice

- Symmetric Matrix-

$$A=\begin{bmatrix}3 & 7 & 8\\ 7 & 4 & 2\\ 8 & 2 & 5\end{bmatrix}$$

```
>>A=[3 7 8; 7 4 2; 8 2 5];      % symmetric matrix
>>B=A'                         % transpose of A
```

Trace of a Matrix

The "trace" Command

The trace of a square matrix is equal to the sum of its diagonal elements. The **trace** command provides the trace of a given matrix.

Practice

- Trance of a Matrix: The "trace" Command-

```
>>% compute the trace of matrix
 >>A=[1 2 3; 4 5 6; 7 8 2];          % define matrix A
 >>trace(A)   <enter>                % calculate the trace of matrix A

ans =
       8
```

Additional Properties of Matrices

1. inv(A*B)=inv(A)*inv(B)
2. (A*B)'=B'*A'
3. (A+B)'=A'+B'

Eigenvalues and Eigenvectors

The "eig" Command

The eigenvalues and eigenvectors of a square matrix can be determined using the function **eig.** There are two flavors of this function, one just finds the eigenvalues, and the other finds both the eigenvalues and the eigenvectors.

Practice

- The "eig" Command-

```
>>A=[1 2; 3 4];                % define matrix A
>>eig(A)   <enter>             % compute the engenvalues of A
```

The same command **eig** can be used to produce both the eigenvalues and eigenvectors. Here is an illustration:

```
>>A=[ 1 2; 3 4];               % define matrix A
>>[V,D]=eig(A)                 % eigenvalues and eigenvectors
```

The eigenvalues of matrix A are stored as the diagonal entries of the diagonal matrix D and the corresponding eigenvectors are stored in columns of matrix V.

Solving Systems of Linear Equations

To solve the linear system Ax=b, where A is known n-by-n matrix, b is known column vector of length n, and x is an unknown column vector of length n.

Ax=b

Multiply each side by the inverse matrix A^{-1}

$\therefore A^{-1}Ax = A^{-1}b$

but $A^{-1}A = I \Rightarrow Ix = A^{-1}b$

$\therefore x = A^{-1}b$

This procedure turns out to be slow. A quicker, and sometimes more accurate method to solve systems of linear equation is based on the Gaussian elimination scheme, where one systematically eliminates one unknown from the equations down to the point where there is only one unknown left. The solution by Gaussian elimination is implemented via the left division operation (backslash),"\":

```
X=A\b
```

There are several direct, eliminating methods for solving systems of n equations and n variables. For large n, the best methods for serving Ax=b are Gaussian elimination and Gauss-Jordan methods. The method of multiplication by A^{-1} is much worse than these and Cramer's rule is the worst of these 3 methods.

Many engineering problems are solved by numerical methods.

Practice

- Solving Systems of Linear Equations-

Solve the following system of linear equation.

$$\begin{cases} x+5y+7z=1 \\ 3x+2y+4z=2 \\ 7x+9y+z=3 \end{cases}$$

This system of linear equation can be written in the form:

$$\begin{bmatrix} 1 & 5 & 7 \\ 3 & 2 & 4 \\ 7 & 9 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 3 & 2 & 4 \\ 7 & 9 & 1 \end{bmatrix}, \qquad w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

First Method

```
>>%Slower procedure
>>A=[ 1 5 7; 3 2 4; 7 9 1];      % specify matrix A
>>b=[1 2 3]';                    % specify the column vector b
>>w=inv(A)*b;                    % compute the solution of the linear system
```

Second Method

```
>>%Faster procedure
>>A=[1 5 7; 3 2 4; 7 9 1];       % specify matrix A
>>b=[1 2 3]';                    % specify the column vector b
>>w=A\b                          % compute the solution of the linear system
```

Cramer's Rule

If Ax=b is a system of n linear equations in n unknowns such that det(A)≠0, then the system has a unique solution. This solution is:

$$x_1 = \frac{\det(A_1)}{\det(A)}$$

$$x_2 = \frac{\det(A_2)}{\det(A)}$$

.

.

.

Where A_j is the matrix obtained by replacing the jth column of A by the column vector b.

```
>>A=[1 5 7; 3 2 4; 7 9 1];
>>b=[1 2 3]';
>>for n=1:3
        D=A
        D(:,n)=b;
        C=D;
        w(n)=det(C)/det(A);
end

>>w=w'
w =

    0.5495
   -0.1099
    0.1429
```