Notes on Linearity, Homogeneity and the Principle of Superposition

Definitions

A *linear differential equation* (ordinary or partial) for an unknown function u (of one or several variables respectively) is one of the form

$$L[u] = f,$$

where f is a known function and L is a *linear differential operator*, i.e. a differential operator satisfying the condition

$$L[c_1u_1 + c_2u_2] = c_1L[u_1] + c_2L[u_2],$$

for any (sufficiently smooth) functions u_1 and u_2 and any real constants c_1 and c_2. The above linear differential equation is said to be *homogeneous* if the function f is identically zero. In other words, a homogeneous linear DE is one of the form $L[u] = 0$ for some linear differential operator L. The classical PDE's of mathematical physics are thus all *homogeneous* PDE's, the linear differential operator L being given for Laplace's equation, the heat equation (with diffusivity α) and the wave equation (with wave velocity c) by $L = \Delta$, $L = \Delta - \frac{1}{\alpha^2}\frac{\partial}{\partial t}$, and $L = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ respectively, where Δ is the Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ or its restriction to one or two dimensions.

A *linear boundary condition* for the linear DE $L[u] = f$ is a condition on u of the form

$$M[u] = g$$

on some surface in 3 dimensions, on some curve in 2 dimensions and at some point(s) in one dimension, g being a known function defined there and M a linear differential operator of order less than that of L. The linear boundary condition is said to be *homogeneous* if the function g vanishes identically.

Principle of Superposition

Suppose we wish to solve a certain linear *boundary value problem*, i.e. to solve a certain linear DE subject to certain linear boundary conditions. Assume further that the linear DE and the linear boundary conditions *are all homogeneous*. Then it is clear from the definition of a linear operator that, if u_1 and u_2 are any two solutions to our problem and c_1 and c_2 are any two real numbers, then $c_1u_1 + c_2u_2$ is also a solution to our problem. By induction, if $u_1, u_2, \ldots, u_n$ are any n solutions to our problem, then so is $c_1u_1 + c_2u_2 + \cdots + c_nu_n$. Subject to considerations of convergence, one can extend this to infinite sums. Thus if the u_k are an infinite sequence of solutions to our problem, then $\sum_{k=1}^{\infty} c_ku_k$ is also a solution, provided the sequence of real numbers c_k converges sufficiently rapidly to zero. This is the form of the *principle of superposition* used in *the method of separation of variables*, in which the u_k are taken to be separable solutions. It usually turns out (although the proof of this is far from trivial) that there are sufficient separable solutions that *every* reasonable solution to our problem can be expressed as an infinite linear combination of them, i.e. may be obtained in the above manner by appropriate choice of coefficients c_k. This is called the property of *completeness*. The principle of superposition is essentially the statement that the solutions to our problem form a vector space (normally infinite-dimensional in the case of a PDE), and completeness means that the separable solutions *span* this space, i.e. that the space has a *basis* of separable solutions.